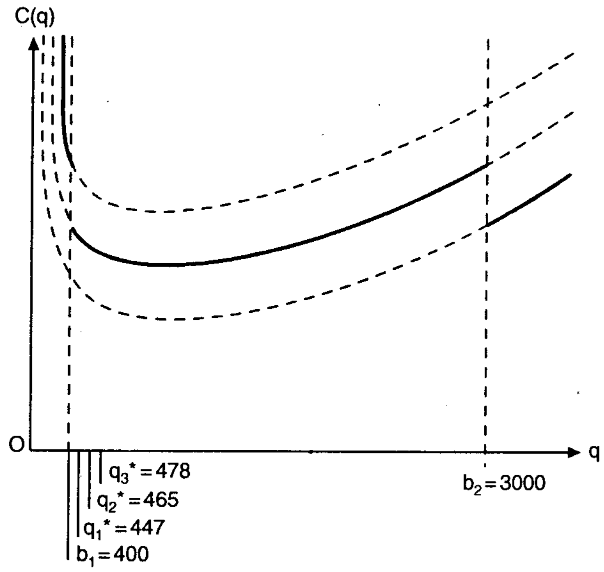

Fig. 20.23. $b^* = 465$

But, from equation (20.91), $C(q_2^*) = C(465) =$ Rs. 1937.00 and $C(3000) =$ Rs. 2020.17.

Since $C(q_2^*) < C(3000)$, the most economic purchase quantity is $q_2^* = 465$.

This situation is shown graphically in *Fig.* 20.23.

Example 44. *This example is same as* ***Example* 42** *except that*

$b_2 = 1500$ (*instead of* 750), $P_2 =$ *Rs.* 9.00 (*instead of Rs.* 8.75).

Solution. Here,

$$q_3^* = \sqrt{\left(\frac{2 \times 100 \times 200}{2 \times 0.02}\right)} = 471 \text{ units, which is less than } 1500,$$

From ***Example* 37,** $q_2^* = 465 < 500$, $q_1^* = 447$.

Compare $C(1500)$, $C(500)$, and $C(q_1^* = 447)$.

We have $C(1500) =$ Rs. 1949.33, and from ***Example* 36.**

$C(500) =$ Rs. 1937.25, $C(q_1^* = 447) =$ Rs. 2090.42.

Hence, in this situation, the optimum purchase quantity is $q^* = 500$. **Ans.**

This situation is shown graphically in *Fig.* 20.24 (on page 727).

Example 45. *This example is same as* ***Example* 42** *except that* $b_1 = 3000$ *(instead of 500) and* $b_2 = 5000$ *(instead of 750).*

Solution. An in ***Example* 42,** $q_3^* = 478$, $q_2^* = 465$, $q_1^* = 447$.

Compare $C(q_1^* = 447)$, $C(3000)$ and $C(5000)$.

From ***Example* 42,** $C(q^* = 447) =$ Rs. 2090.42. and $C(3000) =$ Rs. 2135.17.

Also, $C(5000) =$ Rs. 2192.50 [using equation (20.91)]

Now, compare $C(q_1^* = 447)$, $C(3000)$ and $C(5000)$. It is observed that the optimum purchase quantity is $q^* = 447$. **Ans.**

This situation is shown graphically in *Fig.* 20.25 (on page 727).

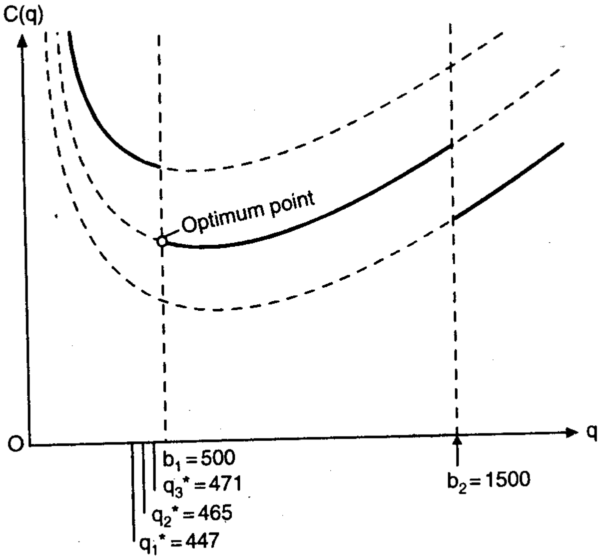

Fig. 20.24. $q^* = 500$

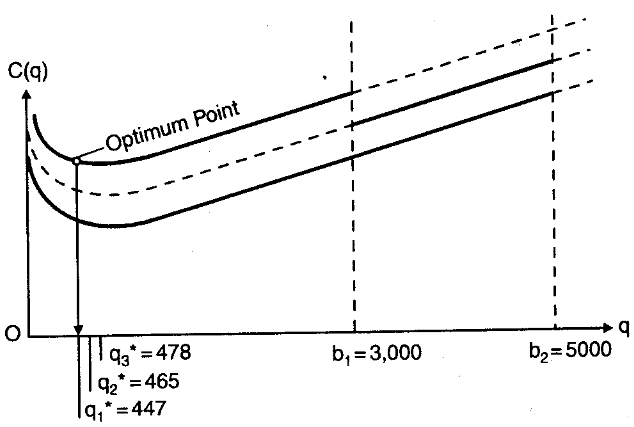

Fig. 20.25. $b^* = 447$

Example 46. *A shopkeeper has a uniform demand of an item at the rate of 50 items per month. He buys from supplier at a cost of Rs. 6 per item and the cost of ordering is Rs. 10 each time. If the stock-holding costs are 20% per year of stock value, how frequently should he replenish his stocks ?*

Now suppose the supplier offers a 5% discount on orders between 200 and 999 items and a 10% discount on orders exceeding or equal to 1000. Can the shopkeeper reduce his costs by taking advantage of either of these discounts ?

Solution. We are given that : $R = 600$ items per year, $C_3 =$ Rs. 10 per order $C_1 =$ Rs. $(6 \times 0.20) =$ Rs. 1.20

$$\therefore \quad q^* = \sqrt{\left(\frac{2C_3R}{C_1}\right)} = \sqrt{\left(\frac{2 \times 600 \times 10}{1.20}\right)} = 100 \text{ items.}$$

From this we observe that the shopkeeper must replenish the inventory after every two months, because 100 items are sufficient to meet the demand of two months only.

The total annual cost includes the fixed cost, set-up cost and stock holding costs. In this case, the fixed cost is Rs. $6 \times 600 =$ Rs. 3600. Also, since each time 100 items are ordered, there will be six orderings throughout the year and hence the replenishment cost is Rs. 60.00.

But, the average inventory throughout the year is $100/2 = 50$ units.

$\therefore$ Average inventory carrying cost $= 50 \times 0.2 \times 6 =$ Rs. 60.00

Hence the total cost = Rs. 3600 + Rs. 60 + Rs. 60 = Rs. 3720.00

In the case of quantity discounts, we have the following formulation :

Quantity	*Unit cost (Rs.)*
$0 \le q_1 < 200$	6.00
$200 \le q_2 < 1000$	5.70 (5% discount)
$1000 \le q_3$	5.40 (10% discount)

Therefore, $$q_3^* = \sqrt{\left(\frac{2C_3R}{P_3I}\right)} = \sqrt{\left(\frac{2 \times 10 \times 600}{(5.40) \times (0.20)}\right)} = 110 \text{ units.}$$

Since $q_3^* < b_2$ (= 1000), we next compute q_2^*.

$$\therefore \quad q_2^* = \sqrt{\left(\frac{2C_3R}{P_2I}\right)} = \sqrt{\left(\frac{2 \times 10 \times 600}{(5.70) \times (0.20)}\right)} = 105 \text{ units.}$$

Again, since $q_2^* < b_1$ (= 200), we next compute q_1^*.

$$\therefore \quad q_1^* = \sqrt{\left(\frac{2C_3R}{P_1I}\right)} = \sqrt{\left(\frac{2 \times 10 \times 600}{(6.00) \times (0.20)}\right)} = 100 \text{ units.}$$

Now compute,

$$C(q_1^*) = 10 \times \frac{600}{100} + 600 \times 6 + (0.20) \times 6 \times \frac{100}{2} = \text{Rs. } 3720$$

$$C(b_1) = 10 \times \frac{600}{200} + 600 \times (5.70) + (0.20) \times (5.70) \times \frac{200}{2} \text{Rs. } 3564.$$

$$C(b_2) = 10 \times \frac{600}{1000} + 600 \times (5.40) + (0.20) \times (5.40) \times \frac{1000}{2} \text{Rs. } 3786.$$

Since $C(b_1) < C(q_1) < C(b_2)$, the optimum purchase quantity is $q^* = b_1 = 200$ units.

Hence the shopkeeper should accept the offer of 5% discount only, because in doing so his net saving during the year would be = Rs. 3720 − Rs. 3564 = Rs. 156.

EXAMINATION PROBLEMS

1. Find the optimal order quantity for a product for which the price breaks are as follows :

Quality :	$0 \le q_1 < 50$	$50 \le q_2 < 100$	$100 \le q_3$
Unit cost :	Rs. 10.00	Rs. 9.00	Rs. 8.00

The monthly demand for the product is 200 units, the cost of storage is 25% of the unit cost and ordering cost is Rs. 20 per order. **[Virbhadrah 2000; Agra 93; Rohil. 92]**

[**Hint.** Here $R = 200$, $P =$ Re. 0.25, $C_3 =$ Rs. 20, $q_3^* = 63$, $q_2^* = 60$, $C(60) > C(100)$.]

[**Ans.** $q^* = b_2 = 100$ units.]

2. Demand for a particular item is 2000 units per year. Unit cost is Rs. 5. Carrying cost is 12 per cent per year and ordering cost is Rs. 10.
(a) Find EOQ. (b) A one per cent discount is offered for this item, if ordered in quantity between 600 to 1000. Should it be taken ? If so, what should be the size of order ?
[**Ans.** (a) $q^* = 237$ units approx. (b) $q^* = 260$ unit approx.]

3. A manufacturer of motors uses Rs. 5 lakhs of values per year. The administrative cost per year is Rs. 500 and the carrying cost is 20% of the average inventory. The company currently has an optimum purchasing policy but has been offered a 0.2 per cent discount if they purchase 5 times per year. Should the offer be accepted ? If not, what counter-offer should be made.

4. A company purchases a certain chemical to use in its process. No shortages are allowed. The demand for the chemical is 1000 kg/month. The cost of one purchase is Rs. 800 and the holding cost for one unit is Rs. 10 per month. The unit cost depends upon the amount purchased. Determine the optimum purchase quantity.

Amount in kg.	*Cost per kg.*
0—449	Rs. 7.50
450 and above	Rs. 7.00

[**Ans.** $q^* = 146$ kg. approx.]

5. **Mujib & Co.** has found that its cost to purchase bars is Rs 40 per order and the carrying charge on average inventory is 10 per cent. They currently purchase Rs. 20,000 of bars a year and make these purchases on an optimum basis. They have been offered a 3 per cent discount on the bars if they purchase quarterly. Should they accept ?

6. A subcontractor has been found who can supply bushings to a power plant manufacturer, who require 83 bushings per day in his assembly operations. No shortages are to be allowed. Procurement cost will be Rs. 90 per purchase order. The cost per unit is a function of the purchase quantity as follows :

Purchase Quantity	:	1—199	200—499	500 or more
Price per bushing (Rs.)	:	115	110	100

The holding cost is Rs. 0.45 per bushing per day. Calculate the least cost purchase quantity for purchasing from the subcontractor.

7. A manufacturer's requirement for an item is 2,000 units per year. Ordering costs are Rs. 100 per order and inventory costs are 16 per cent per year per unit of average inventory. Calculate the economic order quantity. If the price quoted is Rs. 10 each for quantities below 1000 units, Rs. 9.50 for quantities between 1000 and below 2000, and Rs. 9.30 for lots of 2000 or more, compute total ordering cost when ordering is in lots of (i) 500, (ii) 1000, (iii) 2000 units.

8. A manufacturer of engines is required to purchase 4,800 castings per year. The requirement is assumed to be known and fixed. These castings are subject to quantity discounts. The price schedule is as follows :
(i) for less than 500 units Rs. 150.00 per unit.
(ii) for 500 or more but less than 750 units Rs. 140.00 per unit.
(iii) for 750 or more units Rs. 132.00 per unit.
Monthly holding cost expressed as a percentage value of the units = 2%. Ordering cost associated with procurement of items is Rs. 750 per procurement. Find the optimum purchased quantity per procurement.
[**Ans.** q^* = 1,656 units approximately.]

9. A company requires 2500 units of a special type of bolt per year. An offer has been received for supply of this at the rate of 2 Rs./bolt. The supplier has also offered a discount of 3% for purchased lots of size between 1500 and 2499 units. Any order of 2,500 units and above will be supplied at a discount of 5% on the base price. If the inventory carrying cost is 20% a year and the cost of ordering of each lot from the supplier's premises works out to Rs. 20 per lot, is it advantageous to change the order quantity from EOQ and get discount ? If so, what shold be the modified order quantity, EOQ, 1500 or 2500 ?

20.23. PURCHASE-INVENTORY MODEL WITH ANY NUMBER OF PRICE BREAKS

So far the decision rules for determining the optimum purchase quantity are obtained when the purchase cost per item is subject to either one or two price breaks only. Now generalize these decision rules to discuss a purchasing process for any number (n) of price breaks. The situation may be represented as below :

Range of Quantity	:	$0 \le q_1 < b_1$	$b_1 \le q_2 < b_2$	:	$b_{n-1} \le q_n$
Purchase Price per Unit	:	P_1	P_2	:	P_n

Let the optimum purchase quantities for each price be $q_1^*, q_2^*, q_3^*, \ldots, q_n^*$ (see *Fig.* 20.26). Then the following general decision rules apply :

General Decision Rules :

1. ***First, compute*** q_n^*. ***If*** $q_n^* \ge b_{n-1}$, ***then the optimum purchase quantity is*** q_n^*.

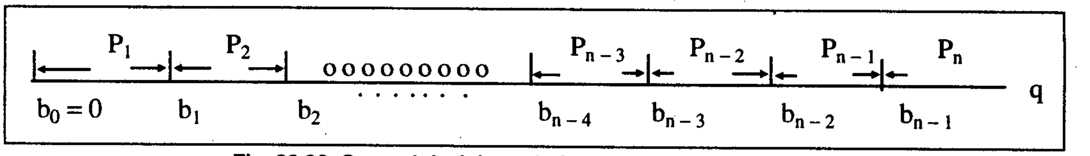

Fig. 20.26. General decision rule for n = 1 price breaks

2. ***If*** $q_n^* < b_{n-1}$, ***then compute*** q^*_{n-1}. ***If*** $q^*_{n-1} > b_{n-2}$, $(b_{n-2} \le q^*_{n-1} < b_{n-1})$, ***then proceed as in the case of one price break, i.e. the optimum purchase quantity is determined by comparing*** $C(q^*_{n-1})$ ***with*** $C(b_{n-1})$.

3. ***If*** $q^*_{n-1} < b_{n-2}$, ***then compute*** q^*_{n-2}. ***If*** $q^*_{n-2} \ge b_{n-3}$, ***then proceed as in the case of two price breaks, i.e. the optimum purchase quantity is determiend by comparing*** $C(q^*_{n-2})$ ***with*** $C(b_{n-2})$ and $C(b_{n-1})$.

4. ***If*** $q^*_{n-2} < b_{n-3}$, ***compute*** q^*_{n-3}. ***If*** $q^*_{n-3} \ge b_{n-4}$, ***then compare*** $C(q^*_{n-3})$ ***with*** $C(b_{n-3})$, $C(b_{n-2})$ and $C(b_{n-1})$.

5. ***Continue in this way until*** $q^*_{n-1} \ge b_{n-(i+1)}$, $[0 \le i \le n-1]$, ***and then compare*** $C(q^*_{n-1})$ ***with***
$$C(b_{n-i}), C(b_{n-i+1}), C(b_{n-i+2}), \ldots, C(b_{n-1}).$$

This procedure will involve a finite number of steps, in fact at the most n, where n denotes the number of price ranges.

Note. It should be noted that $q_1^* < q_2^* < q_3^* < \ldots < q_n^*$.

Q. 1. (a) Describe decision rules for a purchase inventory model, with three price breaks.
(b) Derive an expression for EOQ price discount model. (instantaneous supply with no shortages).
(c) Write a short note on inventory model with price breaks. **[Meerut (OR) 2003]**

20.23–1. Illustrative Examples

'Example 47. *The annual demand for a product is 500 units. The cost of storage per unit per year is 10% of the unit cost. The ordering cost is Rs. 180 for each order. The unit cost depends upon the amount ordered. The range of amount ordered and the unit cost price are as follows :*

Range of amount ordered :	$0 \le q_1 < 500$	$500 \le q_2 < 1500$	$1500 \le q_3 < 3000$	$3000 \le q_4$
Unit cost :	Rs. 25.00	Rs. 24.80	Rs. 24.60	Rs. 24.40

Find the optimal order quantity. **[Meerut (Math.) BP-96; Delhi (OR.) 92]**

Solution. Given that $R = 500$, $C_3 =$ Rs. 180.00, $I = 0.10$, $b_1 = 500$, $b_2 = 1500$, $b_3 = 3000$, $P_1 =$ Rs. 25.00, $P_2 =$ Rs. 24.80, $P_3 =$ Rs. 24.60, $P_4 =$ Rs. 24.40.

Obviously, this problem has three price breaks only.

First compute q_4^* by using the formula (20.92) and obtain

$$q_4^* = \sqrt{\left(\frac{2RC_3}{IP_4}\right)} = \sqrt{\left(\frac{2 \times 500 \times 180}{(0.10) \times 24.40}\right)} = 272 \text{ units.}$$

Since q_4^* $(= 272) < b_3$ $(= 3{,}000)$, $q_4^* = 272$ is not the optimal order quantity. So proceed to compute q_3^* as,

$$q_3^* = \sqrt{\left(\frac{2RC_3}{IP_3}\right)} = \sqrt{\left(\frac{2 \times 500 \times 180}{(0.10) \times 24.60}\right)} = 270 \text{ units,}$$

which does not lie in the range $1500 < q_3 \le 3000$.

Next, compute
$$q_2^* = \sqrt{\left(\frac{2RC_3}{IP_2}\right)} = \sqrt{\left(\frac{2 \times 500 \times 180}{(0.10) \times 24.80}\right)} = 269 \text{ units,}$$

which is less than b_2 and b_1 both. Therefore as per rule, compute q_1^* and then compare the cost $C(q_1^*)$ with $C(b_2)$ and $C(b_1)$.

$$q_1^* = \sqrt{\left(\frac{2RC_3}{IP_1}\right)} = \sqrt{\left(\frac{2 \times 500 \times 180}{(0.10) \times 25}\right)} = 268 \text{ units.}$$

Now,
$$C(q_1^*) = \frac{RC_3}{q_1^*} + RP_1 + \tfrac{1}{2} IP_1 q_1^*$$
$$= \frac{500 \times 180}{268} + 500 \times 25 + \tfrac{1}{2} \times 0.10 \times 25 \times 268 = \text{Rs. } 13170.82;$$
$$C(b_1) = \frac{RC_3}{b_1} + RP_2 + \tfrac{1}{2} IP_2 b_1$$
$$= \frac{500 \times 180}{500} + 500 \times (24.80) + 1/2 \times 0.10 \times (24.80) \times 500 = \text{Rs. } 13200.00$$

and
$$C(b_2) = \frac{RC_3}{b_2} + RP_3 + \tfrac{1}{2} IP_3 b_2$$
$$= \frac{500 \times 180}{1500} + 500 \times (24.60) + \tfrac{1}{2} \times 0.10 \times (24.60) \times 1500 = \text{Rs. } 14205.00.$$

Since $C(q_1^*) < C(b_1) < C(b_2)$, $q_1^* = 268$ units is the optimum order quantity.

Example 48. *Determine an optimal ordering rule for the following case : $R = 6{,}000$ units/month, $C_3 = Rs.\ 50$ per order $I = 0.02$ per month, where*

$P_1 = Rs.\ 1.25$	for	$0 \le q_1 < 100$
$P_2 = Rs.\ 1.20$	for	$100 \le q_2 < 300$
$P_3 = Rs.\ 1.15$	for	$300 \le q_3 < 500$
$P_4 = Rs.\ 1.00$	for	$500 \le q_4 < 1{,}000$
$P_5 = Rs.\ 0.95$	for	$1000 \le q_5 < 2{,}000$
$P_6 = Rs.\ 0.90$	for	$q_6 \ge 2{,}000$

Solution. Since

$$q_6^* = \sqrt{\left(\frac{2C_3R}{IP_6}\right)} = \sqrt{\left(\frac{2 \times 50 \times 6{,}000}{(0.02) \times 0.90}\right)} = \frac{10{,}000}{\sqrt{3}} = 5780 \text{ units (approx.)}.$$

which is greater than b_5 (= 2,000), the optimal order quantity is $q_6^* = 5780$ units. **Ans.**

20.24. DYNAMIC ORDER QUANTITIES (DOQ) SYSTEM

So far, we have discussed that the demand is uniform. But, in actual practice, we come across with non-uniform demand such as having rising or falling trend and/or depicting the seasonal influences.

According to ***M. Kerner***, the following procedure is adopted for scheduling of known but irregular batchwise demand.

Procedure. For each month n (starting with $n = 1$), the condition $n^2R_{n+1} < (C_3/IP)$ is checked, where R_{n+1} is the requirement for the next month, C_3 is the setup cost, and IP is the inventory carrying cost.

So long as the condition is satisfied (*i.e.* answer is 'Yes') n is increased by 1 to take the next month into consideration. But, if as soon as the abvoe condition is not satisfied (*i.e.* answer is 'No'), there is an end of the particular grouping, and the following month is taken as a new month 1 to proceed further in the like manner.

The following example will make the procedure clear.

Example 49. *The monthly requirement schedule for a product is given below :*

Month :	1	2	3	4	5	6	7	8	9	10	11	12
Requirement :	100	150	10	70	90	180	2	98	100	200	140	160

Unit Price (P) = Rs. 15, Set-up cost (C_3) = Rs. 150, and Inventory carrying cost is 30% of the annual average inventory value.

Determine an optimum plan of setups and batch sizes.

Solution. We are given that C_3 = Rs. 150, P = Rs. 15, and I = 30% of annual average inventory value = 0.30/12 = 0.025 monthly.

Hence $$\frac{C_3}{IP} = \frac{150}{0.025 \times 15} = 400.$$

The working procedure may be tabulated as follows :

Month	Requirement	n	n^2R_{n+1}	Is $n^2R_{n+1} < 400$?	Action
1	100	1	150	yes	I set-up
2	150	2	40	yes	I set-up
3	10	3	630	no	I set-up
Size of I set-up	Total = 260				Set-up again in month 4
4	70	1	90	yes	II set-up
5	90	2	720	no	II set-up
Size of II set-up	Total = 160				Set-up again in month 6
6	180	1	2	yes	III set-up

7	2	2	392	yes	III set-up
8	98	3	900	no	III set-up
Size of III set-up	Total = 280				Set-up again in month 9
9	100	1	200	yes	IV set-up
10	200	2	560	no	IV set-up
Size of IV set-up	Total = 300				Set-up again in month 11
11	140	1	160	yes	V set-up
12	160	2	—	—	V set-up

There are five setups with respective sizes of 260, 160, 280, 300, 300. The last setup could possibly take up some of the next years requirements.

Q. Distinguish between static and dynamic inventory models. How can you estimate ordering costs and carrying costs ?

EXAMIANTION PROBLEMS (On Quantity Discounts)

1. Consider an item on which incremental quantity discounts are available. The first 100 units cost Rs. 100 each and additional units cost Rs. 95.00 each. Determine the optimal order quantity Q if $R = 600$ units per year, $I = 0.20$, $C_3 =$ Rs. 50 per set-up.

   [**Hint.** $R = 500$, $C_3 =$ Rs. 50.00, $I =$ Re. 0.20, $P_1 = 100$, $P_2 = 95$. Find $q_2{}^* = 51.3$ units. Since $q_2{}^* < b\,(= 100)$, find $q_1{}^* = 50$ units, $C(q_1{}^*) =$ Rs.51,000, $C(100) = 48{,}700$ $\therefore$ $q^* = b = 100$ units **Ans.**]

2. The annual demand of a product is 10,000 units. Each unit costs Rs. 100 if orders placed in quantities below 2000 units, but for orders of 200 or above the price is Rs. 95. The annual inventory holding cost is 10% of the value of the item and the ordering cost is Rs. 5.00 per order. Find the economic lot size.
   [**Hint.** $R = 10{,}000$, $I =$ Re. 0.10, $C_3 =$ Rs. 5.00.

   Find $q_2{}^* = 103$, $q_1{}^* = 1000$, $C(q_1{}^*) = 1000$, $C(q_1{}^*) =$ Rs. 10,51,000, $C(200) =$ Rs. 10,01,250.]
   [**Ans.** 100 units.]

3. Find the optimal order quantity for a product for which the price breaks are as follows :
   Rs. 20 per unit for $0 \le q < 100$, Rs. 18 per unit for $100 \le q < 200$, Rs. 16 per unit. for $200 \le q$.
   The monthly demand for the product is 400 units. The storage cost is 20% of the unit cost of the product and the cost of ordering is Rs. 25.00 per month.
   [**Hint.** $R = 400$, $I = 0.20$, $C_3 = 25$.

   $q_3{}^* = \sqrt{2C_3R/P_3I} = 79$, $q_2{}^* = 75$, $q_1{}^* = 70$.

   $C(q_1{}^*) = 8283.300$, $C(b_1) = 7480.02$, $C(b_2) = 6770.00$.

   Since $C(200) < C(100) < C(70)$, optimum quantity = 200 units.]

   [**Ans.** $q^* = 200$ units.]

4. Find the optimal purchase quantity q^*, assuming the required data as follows :
   $R = 1000$ units per year, $C_3 =$ Rs. 25 per order, $I = 0.10$; and
   $P_1 =$ Rs. 10.00 for $0 \le q_1 < 150$, $P_2 =$ Rs. 8.00 for $150 \le q_2 < 275$, $P_3 =$ Rs. 7.50 for $q_3 \ge 275$.
   Also, represent this situation graphically.
   [**Ans.** $q^* = 275$ units.]

5. A manufacturer of engines is required to purchase 4,800 castings per year. The requirement is assumed to be known and fixed. These castings are subject to quantity discounts. The price schedule is as follows :

Quantity	*Cost*
for less than 500 units	Rs. 150.00 per unit
for 500 or more but less than 750 units	Rs. 138.75 per unit
for 750 or more units	Rs. 131.25 per unit.

   Monthly holding cost expected as a decimal fraction on the value of the units = 0.02. Set-up cost associated with procurement of purchased item in Rs. 750.00 per procurement.
   Find the optimum purchase quantity per procurement.
   [**Hint.** R = 4,800 castings, $C_3 =$ Rs. 750, $I =$ Re. 0.02. $P = 131.25$,

   $q_3{}^* = \sqrt{2C_3R/P_3I} = 1{,}656$.

   Since $q_3{}^* > b_2$ (750), optimum quantity is 1656 units. **Ans.**]

6. Determine a decision rule, using the basic purchasing Economic Order Quantity model for an annual demand of 20,000 units, ordering cost of Rs. 200 per order and carrying cost of 10 per cent per year. Price per unit is a function of order size. The basic price is Rs. 8.00 per unit. This price is an effect for all orders of less than 5,000 units. Orders for 5,000 or more

but less than 10,000 units may be purchased for Rs. 7.50 per unit. Orders for 10,000 or more units may be purchased for Rs. 7.25 per unit. **[JNTU (MCA III) 2004]**

[**Hint.** Here $R = 20{,}000$, $C_3 =$ Rs. 200, $I = 0.10$, and

$P_1 =$ Rs. 8.00 for $0 \le q_1 < 5{,}000$, $P_2 =$ Rs. 7.50 for $5000 \le q_2 \le 10{,}000$, $P_3 =$ Rs. 7.25 for $q_3 > 10{,}000$.

Use decision rule as given in ***Section* 20.23. (p. 749) Ans.** $q^* = b_2 = 10{,}000$ units.

7. A manufacturer's requirement for an item is 2,000 units per year, the ordering cost is Rs. 10 per year, carrying cost is 16% per unit of average inventory. The purchase price is quoted as Re 1. per unit in quantities below 1000, Rs. 0.95 per unit for purchase in lots of 1000, and Re. 0.93 per unit for a single lot of 2000 units. What purchase policy would you recommend.

8. The annual demand of a product is 10,000 units. Each unit costs Rs. 100 if orders are placed in quantities below 200 units, but for orders of 200 or above the price is Rs. 95. The annual inventory holding cost is 10% of the value of the item and ordering cost is Rs. 5.00 per order, find the economic lot size.

9. Annual demand for an item is 500 units, ordering cost is Rs. 18 per order. Inventory carrying cost is Rs. 15 per rupee per year. Relationship between price and quantity ordered is as follows :

Quantity ordered	:	1—15	16—149	150—549	560 & over
Price per unit (Rs.)	:	10,00	9.00	8.75	8.50.

Specify optimal order quantity and the corresponding price of this item.

[**Ans.** $q^* = 16$ units, $C(q^*) =$ Rs. 6,142.50]

10. A company purchases a certain chemical to use in its process. No shortages are allowed. The demand for the chemical is 1000 kg/month; the cost of one purchase is Rs. 800; and the holding cost for one unit is Rs. 10 per month. The unit cost depends on the amount purchased as given below :

Amount (kg) :	0—249	250—449	450—649	≥ 650
Cost per kg (Rs.) :	8.00	7.50	7.00	6.75

Determine the optimum purchase quantity and the optimum total yearly unit.

[**Ans.** $q^* = 142.857$ kg. approx. $C(q^*) =$ Rs. 19,314 approx.]

EXAMINATIONS REVIEW QUESTIONS

1. (a) Define inventory. What are the advantages and disadvantages of having inventories ?

(b) What are inventories ? Why do we have them ? What are the objectives that should be fulfilled by an inventory control system ? Illustrate with an example.

(c) Discuss the importance of inventory models ?

2. What factors should be considered in arriving at specific value for holding cost ?

3. Describe the basic characteristics of an inventory system. Explain the nature of the probabilistic model in inventory control.

[**Hint.** See Chapter 3.]

4. Most of the bussinessmen view "inventory as a necessary evil." Do you agree with this ?

5. Define lead time. What are the various activities occuring during the lead time ?

6. Compare and contrast the different deterministic models that are commonly used.

7. (i) Define "Economic Batch Quantity". (ii) What are the objectives of inventory control.

8. State and explain the various elements of shortage cost in Inventory Control. What is the difference between "Back Order" and "Lost Sales" ? Will the shortage cost depend on the types of shortages ? Why ?

9. What would happen in Inventory Control, if the following situation exists ?

(i) Suppliers are (or are likely to) facing shortage of materials, and you have to get these materials from the suppliers to run your industry.

(ii) There is a steady supply of the materials, you are looking for in the market, but the demand for your finished product is gradually increasing.

(iii) Suppliers are charging more money for the materials than they should ask for and you need the material in your organisation.

(iv) Demand for the finished product, supply of raw materials for the product and the price of the raw materials are varying very much in the market.

(v) Suppliers of raw materials want to participate in your inventory control decisions.

10. What are the different cost components that effect the management decisions in an inventory management problem ? Write explanatory notes in each cost component.

11. (a) What are safety-stocks ? How would you determine optimum level of safety stocks ?

(b) How would you relate the safety-stocks with service levels in any inventory policy.

12. Explain the following statement :

'Conventional EOQ methods can be replaced by better inventory planning methods these days.'

13. What is the problem of inventory management ?

14. What are the motives of carrying cost ?

15. What is an inventory system ? Give examples of various types of inventory systems.

16. List the classification of the basic components of inventory problems.

17. Describe the basic characteristics of an inventory system.

18. What are the advantages and disadvantages of having inventories ?

19. Why is inventory maintained ? Discuss it and give a classification of inventory models.

20. What are useful models of inventory management and what are the data needed for these models ?

21. What are inventory models ? Enumerate the various types of inventory models. Describe them briefly ?

22. Explain clearly the different costs that are involved in inventory problems with suitable examples. How are they inter-related.

23. Define the terms : Set-up cost, holding cost and shortage cost or penalty cost as applied to an inventory problem.

24. Explain the terms : Lead time, Re-order point, Stock-out cost and Set-up cost. Derive Wilson's *economic lot size formula.*

25. With regard to inventory model explain the following :
(i) Lead time (ii) Probabilitic demand (iii) Safety stock (iv) < R and T > rule.

26. Explain the problem of *inventory control* with deterministic demand.

27. In general inventory problems, discuss the variables that may be controlled and uncontrolled variables.

28. With suitable examples differentiate between the fixed order quantity and the fixed order interval systems of inventory management.

29. Explain the problems involved in inventory control, briefly describing the different types of costs that will generally occur in such problems Also, explain deterministic and stochastic models encountered in inventory control.
[**Hint.** See Chapter 3.]

30. Discuss the economic lot size problem with known uniform demand and finite rate of replenishment.

31. With usual notations derive an expression for the economic order quantity, for a production-inventory situation, with known demand.

32. Describe the economic lot size problem with non-linear restriction and illustrate by an example.

33. Using the following symbols, derive a formula which will solve directly for N :
A = annual requirement in rupee; R = price per unit; P = administrative cost per purchase;
C = carrying charges per unit per year; N = units per economic lot.

34. Derive an expression for "Economic Batch Size" in case of a single item deterministic model with uniform demand and stock replenishment.

35. Derive equations for (i) Economic Lot Size (ELS), and (ii) Time of Production run with finite rate of replenishment.

36. With the help of a Quantity cost curve, explain the significance of Economic Order Quantity.

37. It is said that 'EOQ models, however complex, are restricted by so many assumptions, that they have very limited practical value.' Do you agree with this view ? Illustrate your answer with examples.

38. Derive the Wilson's EOQ formula. What are practical limitations of the EOQ formula ? Discuss its sensitivity.

39. Derive economic lot size formula when no shortages are permissible.

40. For a single item static model give an expression for the total cost per unit time and obtain the optimal order quantity.

41. Give the mathematical formulation of a purchase inventory model and solve it under the assumptions that :
(i) demand is known and uniform (ii) shortages are not permissibe (iii) production supply of commodities is instantaneous.
Derive expressions for the optimal purchase quantity and the optimal total cost.

42. Derive a simple economic lot-size formula and show that

$$\frac{K}{K^*} = \frac{1}{2}\left[\frac{Q^*}{Q} + \frac{Q}{Q^*}\right]$$

where Q^* is the optimum value of Q and K^* is the minimum cost under optimal procurement policy.

43. The purchasing process of a product is guided by the following assumptions :
(i) Demand rate is constant, *(ii)* Demand is both fixed and known, and *(iii)* No shortages are allowed.
If P be the unit purchasing cost, I be the cost of carrying one rupee in inventory value for one year and C_3 be the set-up cost per production run, then determine the optimum values of the total cost, order quantity and scheduling time period.

44. In an inventory problem during each run of time t, inventory builds up at a consant rate of $(r - k)$ units per unit time for time t_1, and during the remaining time t_2 there is no replenishment and the inventory decrease at a constant rate of r units per unit time. C_3 is the set-up cost per run and C_1 is the holding cost per unit per unit time. No shortages are permitted. Show that the minimum lot size Q of inventory is given by

$$\sqrt{2C_3 rk/C_1\,(k-r)}$$

45. Consider a single period inventory model and the demand to be a random variable. Let the initial stock level be given by x units. If the set-up cost is K for ordering, cost of purchase Rs. C per item, selling price Rs. P, and holding cost is Rs. h per item charged at the end of the period; then determine how much to be made available at the beginning of the period.

46. In a manufacturing situation, the production is instantaneous and the demand per year is D. No shortages are allowed. Show that the optimum manufacturing quantity q per run which minimizes the total cost is

$$q = \sqrt{2C_pD/C_k(1 - D/R)}$$

where C_p = set-up cost per run, C_k = holding cost per unit per year
R = manufacturing rate per unit of time ($R > D$)

47. A factory supplies two depots. The holding cost at the factory is C_1 and the set-up cost is C_3. Demand arises only at the depots and is R_1 and R_2 per month. There is shipping set-up cost of C_{31} and C_{32} at each depot and holding costs of C_{11} and C_{12}. Production takes at rate K and shortages are not permitted. Find the optimal ordering policy.

48. Derive an optimal decision-rule for the deterministic demand inventory model with following specifications : Production is instantaneous; total demand R during period T arises at a cosnant rate r. Unit holding cost per unit time is C_1 : shortages are met by back ordering at unit cost C_2 per unit time : set-up cost is C_3 per set-up.

49. Write a shorte note on inventory control.

50. Stating the underlying assumptions, derive economic lot size formula for a manufacturing concern in which the production is instantaneous and the demand is R. C_1, C_2 are the storage and shortages costs per unit product per unit time and C_3 is the set-up cost per production run.

51. A raw material is available locally and can be procured without delay. For a factory, the demand for the raw material is 'a' per month. the order cost is K per order. The storage cost is 'b' per unit of raw material stored per month. The shortages are back-logged. The shortage cost is 'a' per unit short per month. Let Q be the quantity, ordered at intervals of T months so that the cost per month is minimum. Find Q and T.

52. Derive the classical economic lot-size formula : $q = \sqrt{2r(C_3/C_1)}$ based on the deterministic model for one item one level inventory problem and show inventory cycle. How far is it true from reality ? Derive also the minimum value of the average cost per unit time, K_0, in the following form :

$$K_0 = \sqrt{2rc_1c_2c_3(1 - r/k)(c_1 + c_2)^{-1}}$$

Given : q = Quantity produced per production run, r = Demand rate, k = Production rate.
c_1 = Holding cost per unit, per unit time, c_2 = Shortage cost per unit, per unit time, c_3 = Set-up cost per production run.

53. Assume that a company produces n number of items and production of each item is instantaneous. The items are produced in lots. the demand rate for each item is constant and can be assumed to be deterministic. No back orders are to be allowed. The management will not invest in inventory more than M rupees. Formulate the problem mathematically and determine the economic lot size.

54. Discuss two-bin storage model.

55. What are the different costs associated with inventory control problems ? How are they obtained ? Give both the analytical and graphical method of determining Economic order quantity. **[I.C.W.A. (Dec.) 90]**

EXAMINATION REVIEW PROBLEMS

1. A prodcuct is produced at the rate of 50 items per day. The demand occurs at the rate of 30 items per day. Given that (i) set-up cost per order = Rs. 100.00, (ii) holding cost per item per unit time = Re. 0.05.
   (a) Find the economic lot size and associated total cost per cycle assuming that no shortage is allowed.
   (b) Repeat Ex. 1 (a) assuming that the product is purchased, i.e. the stock is replenished instantaneously upon request.
   [Ans. (a) $130\sqrt{3}$; Rs. 11.00, **(b)** $200\sqrt{3}$; Rs. 18.00]

2. The XYZ manufacturing company has determined from an analysis of its accounting and production data for part number 625 that its cost to purchase is Rs. 36 per order and Rs. 2 per part. Its inventory carrying charge is 18% of the average inventory. The demand of this part is 10,000 units per annum. Find :
   (a) What should the Economic Order Quantity be ?
   (b) What is the optimum number of days supply per optimum order ?
   (c) What is the optimum number of orders per year ?
   [Hint. $R = 10{,}000 \times 2 = 20{,}000$ in Rs. C_3 = Rs. 36, C_1 = Re. 0.18. Find q^* using the formula (2.5).]
   [Ans. E.O.Q. = $q^*/2 = 1000\sqrt{2}$ units, $t^* = q^*/R = \sqrt{2}/10$ yrs. = 52 days.]

3. Consider a fixed order size inventory system in which demand pattern is as follows :
   (i) Till any time t in a given time period T, the total demand is $W(t/T)^{1/n}$, where W, T and n are constants.
   (ii) Supply is instantaneous (iii) Inventory carrying cost is C_1 per unit time per unit cost. (iv) Order cost is C_2 per order.
   [Ans. $q^* = C_2/C_1$]

4. A factory uses annually 24,000 units of a raw material which cost Rs. 1.25 per unit. Placing each order costs Rs. 25, and the carrying cost to 6% per year of the average inventory. Find the economic order quantity, and the total inventory cost (including the cost of material). The factory works for 320 days a year. If the procurement time is 10 days and safety stock 450 units, find the re-order point, and the minimum, maximum and average inventories.
   [Hint. See **solved example 27.**]

5. A firm uses every year 12,000 units of a raw material costing Rs. 1.25 per unit. Ordering cost is Rs. 15.00 per order and the holding cost is 5% per year of the average inventory.
   (i) Find the economic order quantity.
   (ii) The firm follows E.O.Q. purchasing policy. It operates for 300 days per year. Procurement time is 14 days, and safety stock is 400 units. Find the re-order point, the maximum inventory and the average inventory.
   [**Hint.** See **solved example 27.**]
6. A contractor has to supply 20,000 units/day. He can produce 30,000 units/day. The cost of holding a unit in stock is Rs. 3 per year and the set-up cost/run is Rs. 50. How frequently, and of what size the production run be made ?
   [**Hint.** $C_1 = 3$, $C_3 = 50$, $R = 20{,}000/\text{day}$, $K = 30{,}000/\text{day}$, use formula (2.43b).]
   [**Ans.** $q^* = 1414$ units, $t^* = 0.7$ day]
7. A manufacturing concern has a fixed cycle demand with a period of one week as follows :

Mon.	Tues.	Wed.	Thus.	Fri.	Sat.	Sun.
9	17	2	0	19	9	14

   Company policy is to maintain constant daily production. The production and shipping departments work seven days a week, and each day's production is available for shipment on the following day.
   If shortage costs four times as much per day as a surplus of the same amount, how much stock should be in hand at the start of business on Monday.
   [**Ans.** 10.]
8. A stock can be replenished instantaneously upon order. Demand occurs at a constant rate of 50 items per unit item. A fixed cost of Rs. 400 is incurred each time an order is placed. Although shortage is allowed, it is the company's policy that the shortage quantity does not exceed 20 units. In the mean time (because of budget limitations) no more than 200 units can be ordered at a time. Find the relationship between the implied holding and shortage cost per unit under optimal conditions.
9. The supplier of an item offer a price discount for 10%, if the order quantity is at least 500 units. The original price of the item is Rs. 6.00. If the annual demand for the item is 3000 units, the order cost per order is Rs. 20/– and the inventory carrying cost is 20% per annum, find the optimum order quantity for the item. Explain all the mathematical formulae used for the purpose.
10. One product produced by FRANCIS TOOLS in a range of compact hand tool. It has a fairly constant demand of 40,000 per year. The hard plastic body is the same for all the tools irrespective of their sizes, but the colour is changed periodically to conform to market requirements. Production-runs in the past have averaged 2,000 tools per day. Set-up costs are estimated at Rs. 350/– per producton-run. A tool that sells for Rs. 250/– at retail outlet is valued at Rs. 90/– when it comes off the production line. Complete carrying costs for production items are set at 20% of the production cost and are based on the average inventory level.
    Determine the following :
    (i) Economic production quantity, (ii) Length of economic production run,
    (iii) Maximum inventory level that can be expected.
11. Define "Economic Production Range". If in the above problem, it is permitted to have 1% increase in the total variable cost, what is the range within which the actual batch size may be chosen.
12. Three items *A*, *B* and *C* are to be purchased with the following characteristics.

Item	*Annual Demand*	*Unit Price (Rs.)*
A	10,000	4,00
B	18,000	3.20
C	25,000	2.50

    The annual holding cost is 25% of the value of the goods stored and the cost of placing an order is taken to be Rs. 2/– per order. Demand is sensibly constant; and two, and only two, orders may be placed a week (Assume a 50 week year). For how much should he order ?
    [**Hint.** Use purchase inventory model of *Sec. 2.17.*]
13. Following data is available in respect of the production of an item :

(i) Preparation cost per set-up	= Rs. 1,000	(v) Cost of carrying inventory 2% per month	
(ii) Rate of produciton per day	= 300 pieces	(vi) No. of working days per year 300	
(iii) Rae of consumption per day	= 100 pieces	(vii) Storage cost negligible	
(iv) Direct cost per piece	= Rs. 1.50		

    (a) Find the batch quantity for minimum total cost per piece.
    (b) For the convenience of scheduling, it was decided that the variable cost per piece may be allowed to increase 5%. What would be the range of production of economic quantity.
14. Determine the optimal order size, reorder point, and expected cost per year under the following conditions.
    (i) Deterministic demand at the rate of 2000 per year, lead time is 1/10yr. and no back orders are permitted.
    (ii) Same as (i), but back orders are permitted with penalty cost of Rs. 160.00/back order.
15. The costs associated with ordering, receiving and inspecting items and placing them in inventories average around Rs. 20.00 per item. Orders arrive in shipments of 150 items. The annual demand is for 1200 items. What is the annual cost ?

16. In determining the Economic Order Quantity, the parameter estimates are :

annual demand = 800, ordering cost = Rs. 20.00, carrying cost = Rs. 9.00.

If the ordering cost and carrying cost contain estimating errors as large as 10%, what will be the potential change in order quantity and in percentage change in annual total cost.

17. An item maintained in inventory has an associated stockout cost of Rs. 10.00 per unit. If the demand during the lead time period turns out to be 114 units and the reorder point has been established at 106 units, what will be the resulting stockout for the inventory cycle.

18. A firm has an annual demand of 1000 units; ordering costs of Rs. 10 per order and carrying cost of Rs. 10 per unit-year. Stockout costs are estimated to be Rs. 40/– each time the firm runs out of stock. How much safety stock is justified by the carrying costs.

19. *XYZ* company buys 18,000 bags of fertilizer each year for its farming operation. It costs the company Rs. 75.00 to place an order, and carrying costs amount to 12.5% of fertilizer's purchasing price of Rs. 60.00 per bag.

(i) Compute the optimal order quantity, Q^*, using the basic fixed order quantity model, and calculate the total annual cost.

(ii) Suppose that 2000 more than the optimal order quantity calculated above is ordered. What is the impact on the total cost.

20. The assembly department of an Engg. Company is using annually 45,000 parts that are manufactured in the fabrication department. The part is valued at Rs. 115.00 per unit, and the total of storage and handling cost is Rs. 18.00 per unit per year. Total production set-up cost is Rs. 45.000. The assembly department requires 180 units/day, while the fabrication department can produce 360 units/day. The plant operates 250 days a year :

(i) Compute the optimal order quantity (ii) How many orders will be placed each year ?

21. A firm orders seven items from the same vendor. The order costs are Rs. 1.50 per purchase order (which is common to many items) and Rs. 0.5 per item (for items included in the order). If carrying costs are 20% per year what is the minimum cost order inteval ? If lead time is 1 month, what is the maximum inventory level for each item ?

The necessary details regarding each item are furnished below,

Item	A	B	C	D	E	F	G
Demand/year in units	*150*	*400*	*125*	*100*	*800*	*70*	*175*
unit cost in Rs.	*1*	*0.5*	*2.0*	*3.0*	*0.5*	*5.0*	*2.0*

22. Consider the following data, regarding fiver prdoucts which a company is manufacturing. There are 250 working days.

Product	Demand units/year	Production cost Rs./unit	Production Rate units/day	Annual holding cost Rs./unit	Set-up cost Rs./set-up
A	5000	6	100	1.60	40.00
B	10,000	5	400	1.40	25.00
C	7,000	3	350	0.60	30.00
D	15,000	4	200	1.15	27.00
E	4,000	6	100	1.65	80.00

It is required to find the following : (i) Best production cycle. (ii) Minimum total annual cost.
Formulate suitably and solve.

23. A Purchase Manager has decided to place order for minimum quantity of 500 Nos. of a particular item in order to get a discount of 10%. From the records it was found out that in the last year 8 orders each of size 200 Nos. have been placed. Given Ordering cost = Rs. 500 per order. Inventory Carrying cost = 40% of the inventory value and the cost per unit = Rs. 400, is the Purchase manager justified in his decison ? What is the effect of his decision to the company ?

[**Hint.** $R = 8 \times 200$, C_3 = Rs. 500, $C_1 = IP = 40\% \times$ Rs. 400.

(i) EOQ = $Q^* = \sqrt{(2C_3R/PI)} = \sqrt{[2 \times 1600 \times 500/(400 \times 0.4)]} = 100$.

For this EOQ, ordering cost = $(R/Q)\ C_3 = (1600/100) \times 500 = 8000$

Inventory carrying cost = $(Q/2)\ \Pi = (100/2) \times 400 \times 0.4 = 8000$

Purchasing cost = $R \times P = 1600 \times 400$ = 6,40,000.

$\therefore$ Total cost = Rs. [8000 + 8000 + 6,40,000] = Rs. 6,56,000 ...(1)

(ii) For the current policy of order size of 200 nos.,

Ordering cost = $(R/Q)\ C_3 = (1600/200) = 500$ = 4,000

Inventory carrying cost = $(Q/2)\ \Pi = (200/2) \times 400 \times 0.4$ = 16,000

Purchase cost = $R \times P = 1600 \times 400$ = 6,40,000

Total cost as per present policy = Rs. [4000 + 16,000 + 6,40,000] = Rs. 6,60,000 ...(2)

(iii) If the manager decides to place an order for a minimum quantity of 500 Nos. to avail a discount of 10%, the cost per item becomes Rs. 360.

Ordering cost = $(R/Q) \times C_3 = (1600/500) \times 500$ = Rs. 1600

Inventory carrying cost = $(Q/2) \times PI = (500/2) \times 360 \times 0.4$ = Rs. 36,000.

Purchase cost = $R \times P = 16000 \times 360$ = Rs. 5,76,000

Total cost for order size of 500 nos. = Rs. [1600 + 36000 + 576000] = Rs. 6,13,600 ...(3)

In view of (i), (ii) and (iii) above, we observe that if the manager decides to place an order for a minimum quantity of 500 items, he will save Rs. [6,60,000 – 6,13,600] = Rs. 46,400 over the policy of placing an order of size of 200 nos. Also, he will save Rs. [6,56,000 – 6,13,600] = Rs. 42,400 over the policy of placing an order of economic size of 100 nos.
Thus the decision of the purchase manager is justified and there will be a saving of Rs. 46,400.]

24. A company, for one of the A class items, placed 6 orders each of size 200 in a year. Given ordering cost = Rs. 600, holding cost = 40%, cost per unit = Rs. 40, find out the loss to the company in not operating scientific inventory policy ? What are your recommendations for the future.
[**Hint.** Here, $R = 6 \times 200 = 1200$ items, C_3 = Rs. 600, $C_1 = PI = 40 \times 0.04$.
Scientific inventory policy (*i.e.* EOQ) is given by the formula :

$$q^* = \sqrt{(2C_3R/PI)} = \sqrt{(2 \times 1200 \times 600/40 \times 0.4)} = 300 \text{ units.}$$

In this case, total cost $= PI\frac{q}{2} + C_3 \times \frac{R}{q} = 40 \times 0.4 \times \frac{300}{2} + 600 \times \frac{1200}{300}$ = Rs. (2400 + 2400) = Rs. 4800.

Total cost under the existing system $= 40 \times 0.4 \times \frac{200}{2} + 600 \times \frac{1200}{200}$ = Rs. (1600 + 3600) = Rs. 5200.

∴ Loss to the company in not operating scientific inventory policy = Rs. (5200 – 4800) = Rs. 400.
Recommendations. Obviously, the company should follow the scientific inventory policy, *i.e.* it should place an order of size 300 units and maintain the reorder level at 300 units, whenever, the stock level touches ROL, place a replenishment order for another 300 units.]

25. The owner of a fleet of wagons has to determine the optimal size of his fleet so that expected cost of maintaining the fleet and of hiring extra wagons in case the demand exceeds the size of his fleet, is minimized. Assuming that the cost of maintaining a wagon is '*a*', the cost of hiring is '*b*' ($b > a$) and p_n is the probability of *n* wagons being demanded on any particular day, determine the optimal size of the fleet.

26. A production process has a steady demand for an imput item. This item is bought from a supplier at a standard cost per item. However, the cost of ordering and receiving delivery of a replenishment is fixed irrespective of the size of the order. The stock holding cost is proportional to the stock value. Introducing appropriate variables, construct a normative model to obtain the Economic order quantity and the corresponding frequency replenishment.

27. Let *D* be the average annual demand in units, *A* be the cost of single order, *a* = cost of holding in stock one unit for one year, *C* = unit price, *Q* = Economic order Quantity. Suppose *d* = discount rate or the price reduction as a percentage of the value of annual demand if order is placed for '*U*' items where $U > Q$. Obtain a rule to find out whether is would pay to buy a larger quantity to take advantage of the discount.

28. Yogesh keeps his inventory in special containers. Each container occupies 10 sq. ft. of store space. Only 5000 sq. ft. of the storage space is available. The annual demand for the inventory item is 9000 containers, priced at Rs. 8 per container. The ordering cost is estimated at Rs. 40/– per order, and the annual carrying costs amount to 25% of the inventory value. Would you recommend to Yogesh to increase his storage space ? If so, how much should be the increase. **[Delhi (M. Com.) 90]**
[**Ans.** Yes, 1000 sq. ft.]

29. A manufacturing company uses certain part at a constant rate of 4000 unit per year. Each unit costs Rs. 2, and the company personnel estimate that it costs Rs. 50 to place an order and that carrying cost of inventory is 20% per year. Find the optimum size of each order and the minimum yearly cost.
[**Ans.** $q^* = 1000$, cost = Rs. 400]

30. For one of the bought-out items, The following are the relevant data :
Ordering cost = Rs. 500, Holding cost = 40%, Cost per item = Rs. 100, Annual demand = 1,000.
The purchase manager placed five orders of equal quantity in one year, in order to avail the discount of 5% on the cost of the items. Work out the gain or loss to the organisation due to his ordering policy for this item. **[I.C.W.A. (Dec.) 90]**

31. A trading company buys and sells 10,000 bottles of pain-balm every year. The cost per bottle is Rs. 2 and the company's cost of placing an order for the pain-balm is Rs. 100. The company's standard annual rate of return on working capital funds is 15%. The cost for physical storage of the pain-balm is fixed. Determine :
(a) Optimal order quantity and inventory cycle duration (b) How many orders should be placed each year (c) Find the total relevent annual inventory cost for the pain-balm. **[Karnataka B.E. (CSE/ISE) 93]**

32. (a) What factors are considered in arriving at the specific values for average procurement cost per order and the inventory carrying cost ?
(b) A health centre requires 2000 units of a perticular drug per month. Each unit of drug costs Rs. 3 and the average procurement cost per order is estimated to be at Rs. 150. If the inventory carrying cost is 30% of the average inventory valuation per annum, what quantity of the drug should be ordered ? If the procurement lead time is 6 days, what should the re-order level be ? **[Delhi MBA (HCA) Dec. 94]**

33. If demand note of a product is 500 units per year, the set-up cost is Rs. 800 per lot, *i* is the annual cost of carrying one rupee in the inventory value, and the prices of the unit product are given as :

$$0 \le q < 500, 500 \le q < 1000, 1000 \le q < 2000$$
$$p = \cdot 30 \text{ Rs.}, p = \cdot 29 \text{ Rs.}, p = \cdot 28 \text{ Rs.}$$

Find the optimal lot size. **[Meerut (Maths.) Jan. 98 BP]**
[**Hint.** Proceed as Ex. 41, page 744]

34. A publishing house purchases 2,000 units of a particular item per year at unit cost of Rs. 20, ordering cost per order is Rs. 50 and inventory carrying cost is 25% of the unit cost per year. Find the optimum order quantity and total cost per

year for the item. If 3% discount is offered by the supplier for the purchase in lots of 1000 units or more, should the publishing house accept the proposal.
[JNTU (B. Tech.) 2003]

35. A motor manufacturing Co. purchases 18,000 items of certain motor part for its annual requirement, ordering one-month usage at a time. Each spare costs Rs. 20, the ordering cost per order is Rs. 15 and carrying charges are 15% of the unit item cost per year. Make a more economical purchasing policy. What is the saving by the new policy ?
[JNTU (Mech. & Prod.) 2004]

36. An engineering firm has determined from the analysis of past accounting and production data that part number 607 has ordering cost of Rs. 350 per order and its cost Rs. 22 per part. The inventory carrying cost is 15% of the cost of the item per year. The firm currently purchases Rs. 2,20,000 worth of this part every year. Determine the economic order quantity. What is the time between two orders ? What is optimum number of orders per year to minimize the cost for the firm ?
[JNTU (Mech. & Prod.) May 2004]

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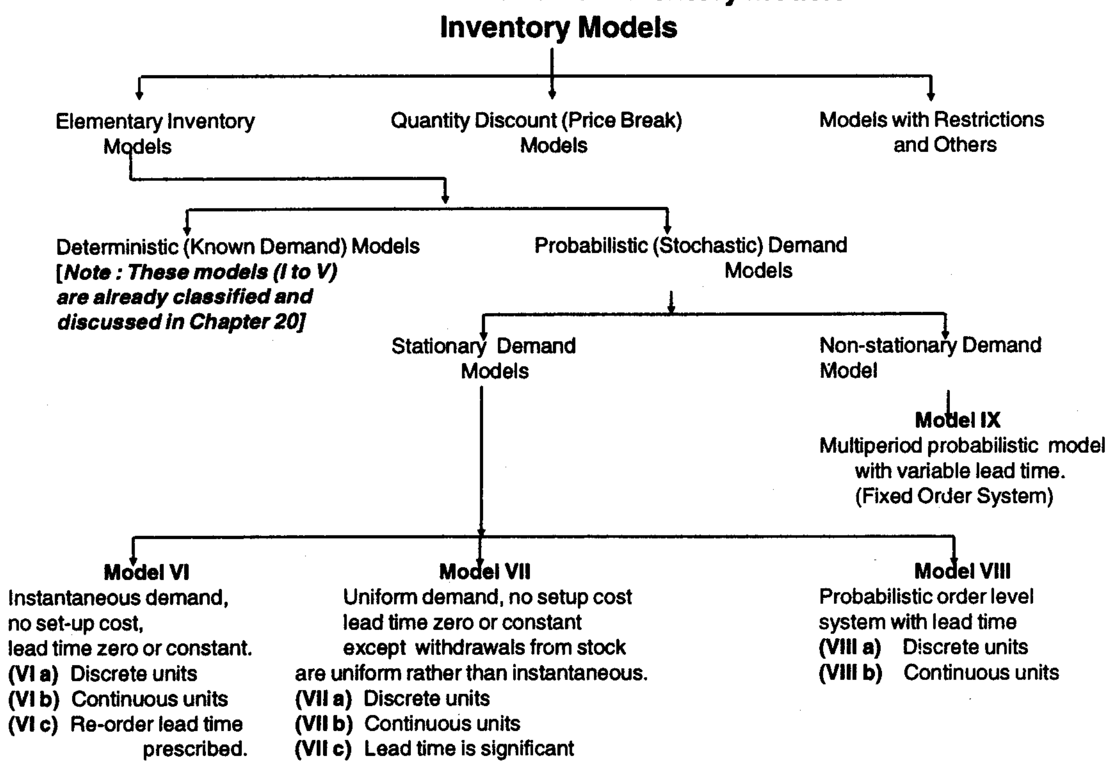
Inventory/Production Management-II
(Stochastic Inventory Models & ABC Analysis)

21.1. INTRODUCTION

The inventory models discussed in the previous chapter seems to be unrealistic because, in practical situations it seldom happens that future demand is known exactly. But, the probability distribution of future demand can be determined by using well-known statistical techniques. Since the probabilistic demand may also be stationary or non-stationary, the probabilistic models can be further classified according to these characteristics as follows :

Further Classification of Inventory Models



In probabilistic systems, we minimize the total *expected costs* rather than *actual costs.*

21.2. DETERMINATION OF SAFETY-STOCK UNDER NORMAL DISTRIBUTION OF DEMAND (Lead Time is Fixed)

Let d = average demand during lead time
B = buffer stock (safety-stock)
$\underline{X}$ = Random demand during lead time
$\bar{X}$ = expected demand during lead time
σ_x = standard deviation of X

In fact, safety stock depends on the service level required by the organisation. To explain this point, suppose the demand during lead time possesses a *normal distribution* with mean $\bar{X}$ and standard deviation σ_x. This distribution is shown in the following figure.

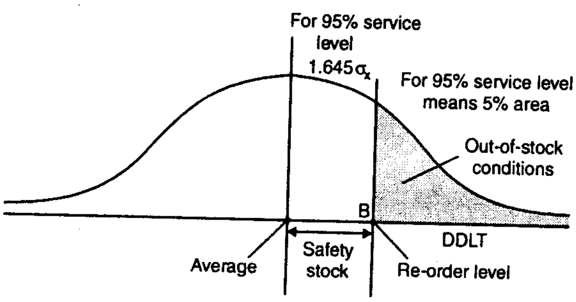


Fig. 21.1.
Distribution of Demand During Lead Time (DDLT) and Safety Stock.

The 95% service level means that the chances of demand during lead time exceeding the re-order point quantity will be 5% only. Thus the value of B (buffer stock) on the normal curve can be determined using normal distribution tables. For 95%, the point B on the curve will be 1.645 standard deviation away from the mean $\bar{X}$.

Therefore, *safety stock* = $1.645\ \sigma_x$, *re-order point* = $\bar{X} + 1.645\ \sigma_x = Ld + 1.645\ \sigma_x$,

Following examples are presented to make the procedure clear.

Illustrative Examples

Example 1. *XYZ company wants to provide a 95 per cent service level to its customers. Using the past history of demand, the following data is available :*

Daily demand follow normal distribution with average daily demand of 20 units and the standard deviation of 5 units. The lead time for procurement is 4 days. The cost of placing an order is Rs. 10 and the inventory carrying cost is Re. 1 per unit per year. There are no stockout costs, and unfilled orders are supplied after the items are received. What should be the inventory policy for the comapny ?

Solution. The annual demand = 20 × 365 = 7300 units. (daily demand × no. of days in a year)

$$\therefore \quad q^* = \sqrt{\left(\frac{2DC_3}{C_1}\right)} = \sqrt{\frac{2 \times 7300 \times 10}{1}} = 381 \text{ units.}$$

Expected demand during lead time : μ = average demand per day × lead time = $d \times L$ = 20 × 4 = 80 units.

Variance of demand during lead time (DDLT) :

$$\sigma^2 = \text{sum of variances of demand for each day with lead time} = \sum_{i=1}^{4} (\sigma_i)^2 = 4\,(5)^2 = 100.$$

$\therefore$ S.D. of demand during lead time,
$\sigma = \sqrt{100} = 10$ units.

This distribution is shown in the following figure.

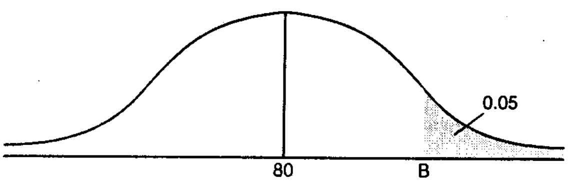


Fig. 21.2

For 95% service level, from normal tables, point B will be 1.645 standard deviation away from the mean. Therefore, we get *safety stock* = $1.645 \times \sigma_x = 1.645 \times 10 \cong$ 16 units.

re-order point = average demand during lead time + safety stock = 80 + 16 = 96 units.

Re-order level X, corresponding to a service level of 95% can also be determined as follows :

$$Z = \frac{X - \mu}{\sigma} \quad (Z = 1.645 \text{ for } 95\% \text{ service level})$$

$$\therefore \quad \frac{X - 80}{10} = 1.645 \quad \text{or} \quad X = 80 + 10 \times 1.645 \cong 96 \text{ units.}$$

Example 2. *The demand per month for a product is distributed normally with a mean of 100 and standard deviation 25. The lead time distribution is given below. What service level will be afforded by a reorder level of 500 units ?*

Lead Time (Months) :	1	2	3	4	5
Probability :	0.10	0.20	0.40	0.20	0.10

Solution. It is given that the demand is distributed normally with :
Mean $(\bar{D})$ = 100 units; *S.D.* (σ_d) = 25 units; *lead time* (L) = 1, 2, 3, 4 and 5; and
re-order level (M) = 500 units.

We shall use iterative method of computing service level for the reorder level policy when the demand per unit time is distributed normally and the distribution of lead times is known. With the help of given information, we can calculate the value of normal variate K for each and every lead time given in the question by the formula :

$$K = \frac{M - L\bar{D}}{\sigma_d \sqrt{L}}$$

Calculations by Iterative Method

Lead Time	Value of K when $M = 500$	Probability of not running out of stock corresponding to the value of K calculated	Probability of this particular lead time occurring	Conditional probability of not running out of stock, %
		(1)	(2)	(1) × (2)
1.	$\frac{500 - 100 \times 1}{25\sqrt{1}} = 16.0$	100	0.10	10.0
2.	$\frac{500 - 100 \times 2}{25\sqrt{2}} = 8.49$	100	0.20	20.0
3.	$\frac{500 - 100 \times 3}{25\sqrt{3}} = 4.49$	100	0.40	40.0
4.	$\frac{500 - 100 \times 4}{25\sqrt{4}} = 2.00$	97.7	0.20	19.5
5.	$\frac{500 - 100 \times 5}{25\sqrt{5}} = 0.00$	50.0	0.10	5.0
Total				94.5

Hence, a reorder level of 500 units will give 94.5 per cent service level.

EXAMINATION PROBLEM

Q. Daily demand for a product is normally distributed with mean, 60 units and a standard deviation of 6 units. The lead time is constant at 9 days. The cost of placing an order is Rs. 200 and the annual holding costs are 20% of the unit price of Rs. 50. A 95% service level is desired for the customers, who place orders during the reorder period. Determine the order quantity and the reorder level for the item in question, assuming that there are 300 working days during a year.
[Delhi (M.B.A.) 95]

I – Stochastic Inventory Models

21.3. INSTANTANEOUS DEMAND, NO SET-UP COST MODEL

It will be assumed here that all demand distributions are stationary and independent over time.

21.3–1. Model VI(a) : Discrete Case

Find the optimum order level z which minimizes the total expected cost under the following assumptions :

(i) ***t is the constant interval between orders (t may also be considered as unity, e.g. daily, weekly, monthly, etc.)***

(ii) ***z is the stock (in discrete units) at the beginning of each period t.***

(iii) ***d is the estimated (random) demand at a discontinuous rate with probability p(d), i.e. demand d arises at each interval t with probability p(d),***

(iv) ***C_1 is the holding cost per item per t time unit,***

(v) ***C_2 is the shortage cost per item per t time unit,***

(vi) ***lead time is zero.***

(vii) ***demand is instantaneous.*** **[Meerut (M.Sc.) 96]**

Solution. In the model with instantaneous demand, it is assumed that the total demand is filled at the beginning of the period. Thus, depending on the amount d demanded, the inventory position just after the demand occurs may be either *positive (surplus)* or *negative (shortage).* These two cases are shown in *Fig.* 21.3. (a) and *Fig.* 21.3. (b).

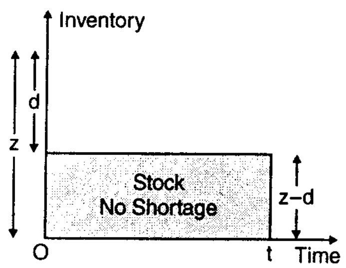


Fig. 21.3 (a). When d < z (over-supply)

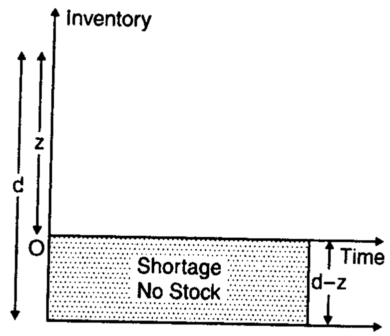


Fig. 21.3 (b). When d > z (under-supply)

Case 1. ***When demand d does not exceed the stock z, i.e. d ≤ z.***

In this case, only the cost of holding inventory due to over-supply is involved as shown in *Fig.* 21.3 (a) and there are no shortages as all customer's demand is satisfied.

Therefore, holding cost per unit time becomes $= \begin{cases} (z-d)\,C_1, & \text{for } d \le z, \\ C_1 \times 0, & \text{for } d > z, \text{ (no stock)} \end{cases}$

Case 2. ***When demand d exceeds the stock z, i.e. d > z.***

In this case, the customer's demand is not satisfied at all, therefore

shortage cost per unit time becomes $= \begin{cases} C_2 \times 0, & \text{for } d \le z, \text{ (no shortage)} \\ (d-z)\,C_2, & \text{for } d > z, \end{cases}$

To get the expected cost, we have to multiply the cost by given probability $p(d)$. Further, to get the total expected cost, we must sum over all the expected costs, *i.e.* the costs associated with each possible value of d. Thus, the total expected cost per unit time is given by the cost equation

$$C(z) = \underbrace{\sum_{d=0}^{z} (z-d)\,C_1\,p(d) + \sum_{d=z+1}^{\infty} C_1 \,.\, 0 \,.\, p(d)}_{\textbf{Case I : } \textit{for demand} \le \textit{stock}} + \underbrace{\sum_{d=0}^{z} C_2 \,.\, 0 \,.\, p(d) + \sum_{d=z+1}^{\infty} (d-z)\,C_2\,p(d)}_{\textbf{Case II : } \textit{for demand} > \textit{stock}}$$

or $$C(z) = \sum_{d=0}^{z} (z-d)\,C_1\,p(d) + \sum_{d=z+1}^{\infty} (d-z)\,C_2\,p(d) \text{ (cost equation)} \qquad ...(21.1)$$

For minimum of $C(z)$, the following condition must be satisfied.

$$\Delta C(z-1) < 0 < \Delta C(z) \quad \text{[see \textit{Unit I, Chapter 2, Finite Differences}]} \qquad ...(21.2)$$

But, we can difference (21.1) under the summation sign if, for $d = z + 1$, the following condition is satisfied;

$$C_1\,[(z+1)-d\,]\,p(d) \equiv C_2\,[d-(z+1)]\,p(d),$$

This is obviously satisfied here.

Now, taking first difference on both sides of equation (21.1), we obtain

$$\Delta C(z) = C_1 \sum_{d=0}^{z} [\{(z+1)-d\} - (z-d)]\, p(d) + C_2 \sum_{d=z+1}^{\infty} [\{d-(z+1)\} - (d-z)]\, p(d)$$

$$= C_1 \sum_{d=0}^{z} p(d) - C_2 \sum_{d=z+1}^{\infty} p(d) = C_1 \sum_{d=0}^{z} p(d) - C_2 \left[\sum_{d=0}^{\infty} p(d) - \sum_{d=0}^{z} p(d) \right]$$

$$= (C_1 + C_2) \sum_{d=0}^{z} p(d) - C_2, \quad \text{because} \left(\sum_{d=0}^{\infty} p(d) = 1 \right)$$

Using condition (21.2) for minimum, we must have $\Delta C(z) > 0$.

$$\therefore \quad (C_1 + C_2) \sum_{d=0}^{z} p(d) - C_2 > 0 \quad \text{or} \quad \sum_{d=0}^{z} p(d) > \frac{C_2}{C_1 + C_2} \qquad \ldots(21.3)$$

Thus, the optimum value of stock level z can be obtained by the relationship :

$$\sum_{d=0}^{z-1} p(d) < \frac{C_2}{C_1 + C_2} < \sum_{d=0}^{z} p(d) \quad \text{[Kanpur 97; Shivaji 85]} \qquad \ldots(21.4)$$

An Important Note :

It should be noted that—if z_0 is such that

$$\sum_{d=0}^{z_0 - 1} p(d) < \frac{C_2}{C_1 + C_2} = \sum_{d=0}^{z_0} p(d),$$

then equation (21.1) leads to $C(z_0 + 1) = C(z_0)$. So the optimum value of z is either z_0 or $(z_0 + 1)$,

Equivalently, if z_0 is such that

$$\sum_{d=0}^{z_0 - 1} p(d) = \frac{C_2}{C_1 + C_2} < \sum_{d=0}^{z_0} p(d),$$

then equation (21.1) leads to $C(z_0 - 1) = C(z_0)$. so the optimum value of z is either $(z_0 - 1)$ or z_0.

Q. 1. Formulate and solve a discrete stochastic model for a single product with lead time zero. The storage and shortage costs are independent of time. Set-up cost is constant. **[Raj. (M.Phil) 93]**

2. Show that for a probabilistic inventory model with instantaneous demand and no set-up cost, the optimum stock level z can be obtained by the relationship

$$\sum_{d=0}^{z} p(d) > \frac{C_2}{C_1 + C_2} > \sum_{d=0}^{z-1} p(d).$$

[Kanpur M.Sc. (Math.) 97; Garhwal M.Sc. (Stat.) 95, 93]

3. Discuss the probabilistic inventory model with instantaneous demand and no set-up cost.

4. Explain and solve an inventory model with instantaneous discrete random demand with no set-up cost : **[Garhwal M.Sc. (Math.) 94]**

21.3-2. Illustrative Examples

Example 3. Newspaper-Boy Problem. *A newspaper-boy buys papers for Rs. 2.60 each and sells them for Rs. 3.60 each. He cannot return unsold newspapers. Daily demand has the following distribution :*

No. of customers :	23	24	25	26	27	28	29	30	31	32
Probability :	.01	.03	.06	.10	.20	.25	.15	.10	.05	.05

If each day's demand is independent of the previous day's, how many papers should he order each day ?

[Meerut (Eco) 2005, (Maths) 2003, 93, (Stat.) 98; JNTU (B. Tech.) 97; Garhwal M.Sc. (Math.) 95, (Stat.) 91; I.C.W.A. (June) 90]

Solution. First we discuss this problem in general. We consider the situation of a newspaper boy who must decide how many newspapers to order so as to maximize his expected profit. He buys a certain number of newspapers each day and sells some or all of them. He makes a profit on each one he sells. He cannot return unsold newspapers. The number of people who want newspapers varies from day to day, but the probability that any specified number of newspapers will be sold on a particular day is known.

Let z = the number of newspapers ordered per day,

d = *the demand that is, the number that could be sold per day*, if $z \geq d$,

$p(d)$ = *the probability that the* demand will be equal to d on a randomly selected day.

c_1 = cost per newspaper, and

c_2 = selling price per paper.

If the damand d exceeds z (the number ordered) his profit would become equal to $(c_2 - c_1)\, z$, and no newspaper will be left unsold.

On the other hand, if demand d does not exceed z (the number ordered), his profit becomes

$$= (c_2 - c_1)\, d - (z - d)\, c_1$$

[because, out of z number of papers only d papers are sold and thus $(z - d)$ number of papers remain unsold causing a loss of the purchase price $(z - d)\, c_1$ of unsold papers $(z - d)$].

Then the expected net profit per day can be expressed as

$$P(z) = \underbrace{\sum_{d=0}^{z} (c_2 d - c_1 z)\, p(d)}_{\textbf{(for } d \leq z)} + \underbrace{\sum_{d=z+1}^{\infty} (c_2 - c_1)\, z\, p(d)}_{\textbf{(for } d > z)}$$

Using finite differences we know that the conditions for maximum of $P(z)$ are

$$\Delta P(z-1) > 0 > \Delta P(z).$$

Also, we can difference under the summation sign if, for $d = z + 1$,

$$[c_2 d - c_1 (z+1)]\, p(d) \equiv (c_2 - c_1)(z+1)\, p(d)$$

i.e.
$$[c_2 (z+1) - c_1 (z+1)]\, p(d) \equiv (c_2 - c_1)(z+1)\, p(d)$$

which is obviously true here.

Thus, differencing under the summation sign, we get

$$\Delta P(z) = \sum_{d=0}^{z} [\{c_2 d - c_1 (z+1)\} - (c_2 d - c_1 z)]\, p(d) + \sum_{d=z+1}^{\infty} (c_2 - c_1)\, [(z+1) - z]\, p(d)$$

$$= - c_1 \sum_{d=0}^{z} p(d) + (c_2 - c_1) \sum_{d=z+1}^{\infty} p(d)$$

$$= - c_1 \sum_{d=0}^{z} p(d) + (c_2 - c_1) \left[\sum_{d=0}^{\infty} p(d) - \sum_{d=0}^{z} p(d) \right]$$

$$= - c_2 \sum_{d=0}^{z} p(d) + (c_2 - c_1), \left(\text{using the fact } \sum_{d=0}^{\infty} p(d) = 1 \right)$$

Since for maximum $P(z)$, $\Delta P(z) < 0$, we have

$$\therefore \qquad - c_2 \sum_{d=0}^{z} p(d) + (c_2 - c_1) < 0$$

or

$$\sum_{d=0}^{z} p(d) > \frac{(c_2 - c_1)}{c_2} \qquad \text{...(21.5)}$$

In this problem, we are given c_1 = Rs.2.60, c_2 = Rs. 3.60, the lower limit for demand d is 23 and upper limit for demand d is 32. Therefore, substituting these numerical values in (21.5), we get

$$\sum_{d=23}^{z} p(d) > \frac{3.60 - 2.60}{3.60} \quad \text{or} \quad \sum_{d=23}^{z} p(d) > 0.28 \text{ (approx.)}$$

We can easily verify that this inequality holds for $z = 27$.

$$\left[\because \sum_{d=23}^{27} p(d) = p(23) + p(24) + p(25) + p(26) + p(27) = .01 + .03 + .06 + .10 + .20 = 0.40 > 0.28, \text{ and similarly, } \sum_{d=23}^{26} p(d) < 0.28\right]$$

Alternative. Letting C_1 = holding cost/paper, C_2 = shortage cost/paper and using the result (21.3) of ***Model VI (a)***, we have

$$\sum_{d=23}^{z} p(d) > \frac{C_2}{C_1 + C_2}.$$

But, $C_1 = c_1$ = Rs. 2.60, and $C_2 = c_2 - c_1$ = Rs.3.60 – Rs. 2.60 = Re. 1.00

Hence, $\sum_{d=23}^{z} p(d) > \frac{1}{3.60}$ which gives us the value of $z = 27$ as above.

Example 4. *Some of the spare parts of a ship cost Rs.1,00,000 each. These spare parts can only be ordered together with the ship. If not ordered at the time when the ship is constructed, these parts cannot be available on need. Suppose that a loss of Rs. 10,000,000 is suffered for each spare part that is needed when none is available in the stock. Further, suppose that probabilities that the spare part will be needed as replacement during the life-term of the class of the ship discussed are :*

Spare parts required	:	0	1	2	3	4	5 *or more*
Probabilty	:	0.9488	0.0400	0.0100	0.0010	0.0002	0.0000

How many spare part should be procured ?

Solution. Since C_1 = Rs. 1,00,000 each part, C_2 = Rs. 10,000,000 each part, then using the result (21.3.), we have $\sum_{d=0}^{z} p(d) > 0.99$ which gives $z = 2$. **Ans.**

21.3-3. Model VI(b) : Continuous Case

[JNTU (B. Tech.) 2003]

This model is the same as *Model VI (a)* except that the stock levels are now assumed to be continuous (rather than discrete) quantities. So instead of probability $p(d)$ we shall have $f(x)\,dx$, where $f(x)$ is the probability density function [see **Unit 3 : Prob. Theory**].

Solution. Let $\int_{x_1}^{x_2} f(x)\,dx$ = the probability of an order within the range x_1 to x_2.

Now, the cost equation for this model will be similar to that which is just derived for ***Model VI (a).*** Only $p(d)$ is replaced by $f(x)\,dx$ and summation Σ is replaced by the sign of integration ($\int$). Then the cost equation for this model becomes :

$$C(z) = C_1 \int_0^z (z - x) f(x)\,dx + C_2 \int_z^{\infty} (x - z) f(x)\,dx. \qquad ...(21.6.)$$

The optimal value of z is obtained by equating to zero the first derivative of $C(z)$, *i.e.*

$$dC/dz = 0. \qquad ...(21.7)$$

We also know the formula that—

if $C(z) = \int_{a(z)}^{b(z)} f(x, z)\,dx$, then

$$\frac{dC}{dz} = \int_{a(z)}^{b(z)} \frac{\partial f(x, z)}{\partial z}\,dx + \left[f(x, z)\frac{dx}{dz}\right]_{a(z)}^{b(z)} \qquad ...(21.7)'$$

Therefore, differentiating the equation (21.6), we get

$$\frac{dC}{dz} = C_1 \int_{x=0}^{z} (1 - 0) f(x)\,dx + C_1\left[(z - x) f(x)\frac{dx}{dz}\right]_{x=0}^{z} + C_2 \int_z^{\infty} (0 - 1) f(x)\,dx + C_2\left[(x - z) f(x)\frac{dx}{dz}\right]_z^{\infty}$$

$$= C_1 \int_0^z f(x)\,dx - C_2 \int_z^{\infty} f(x)\,dx \quad = C_1 \int_0^z f(x)\,dx - C_2\left[\int_0^{\infty} f(x)\,dx - \int_0^z f(x)\,dx\right]$$

$$\therefore \quad \frac{dC}{dz} = (C_1 + C_2) \int_0^z f(x)\,dx - C_2 \qquad \left(\text{since} \int_0^\infty f(x)\,dx = 1\right)$$

But, from equation (21.7), we have

$$(C_1 + C_2) \int_0^z f(x)\,dx - C_2 = 0 \quad \text{or} \quad \int_0^z f(x)\,dx = \frac{C_2}{C_1 + C_2}. \qquad ..(21.8)$$

Also,

$$\frac{d^2C}{dz^2} = (C_1 + C_2)\left[f(x) \cdot \frac{dx}{dz}\right]_0^z = (C_1 + C_2) f(z) > 0 \qquad [\text{since} f(z) > 0 \text{ and } C_1, C_2 \text{ are not zero}].$$

Hence, we can get optimum value of z satisfying (21.8) for which the total expected cost C is minimum.

Q. 1. Write a note on Newspaper-Boy problem.

2. Consider an inventory model in which the cost of holding one unit in the inventory for a specified period is C_1 and the cost of shortage per unit is C_2. Also suppose the demand follows a known continuous probability distribution. Determine, the optimum inventory level in the begining in of the period.

3. (a) A shop-keeper has to decide how much quantity of bread he should stock every week. The quantity of bread demanded in any week is assumed to be a continuous random variable with a given function $f(x)$. Let A be the unit cost of purchasing bread, B unit sale price, C refund on unit stale bread and finally D is the unit penalty cost. Find the optimum quantity of bread to be stocked.

(b) If $A = 8$, $B = 20$, $C = 2$ and $D = 5$, and the demand is regular between 1000 and 2000, show that the optimum quantity is approximately 1739 breads. **[Meerut (Stat.) 90]**

4. Explain and solve that general single product model of profit maximization with time independent cost. **[Gahrwal M.Sc. (Math.) 95]**

21.3–4. Illustrative Examples

Example 5. *A baking company sells cake by the kg weight. It makes a profit of Rs. 5.00 a kg. on each kg sold on the day it is baked. It disposes of all cake not sold on the date it is baked at a loss of Rs. 1.20 a kg. If demand is known to be rectangular between 2000 and 3000 kg., determine the optimal daily amount baked.* **[Agra 99, 98; Meerut 98]**

Solution. Let

$c_1 =$ the profit per kg.

$c_2 =$ the loss per kg. for unsold cake,

$x =$ the demand which is continuous with probability density $f(x)$,

$\int_{x_1}^{x_2} f(x)\,dx =$ the probability of an order within the range x_1 to x_2,

and $z =$ the stock level.

There are two possibilities in this problem :

(i) **If $x \le z$,** that is, the demand x does not exceed the stock z in hand, then clearly the demand x is satisfied leaving $(z - x)$ quantity unsold which is returned with a loss of c_2 per kg.

Since profit is $c_1 x$ and loss is $c_2(z - x)$, net profit becomes $= c_1 x - c_2(z - x) = (c_1 + c_2)x - c_2 z$.

(ii) **If $x > z$,** that is, demand x exceeds the stock z in hand, then nothing remains unsold. Therefore, net profit becomes $= c_1 z$.

Thus, total expected profit within the range x_1 to x_2 is given by

$$P(z) = \underset{x \le z}{\int_{x_1}^{z} [(c_1 + c_2)x - c_2 z] f(x)\,dx} + \underset{x > z}{\int_z^{x_2} c_1 z f(x)\,dx}, \qquad ...(21.9)$$

$$= P_1(z) + P_2(z), \text{ say for convenience.}$$

Now, for maximum $P(z)$, we must have

$$\frac{dP(z)}{dz} = \frac{dP_1(z)}{dz} + \frac{dP_2(z)}{dz} = 0 \qquad ...(21.10)$$

Since $P_1(z) = \int_{x_1}^{z} [(c_1 + c_2)\, x - c_2 z] f(x)\, dx$, so differentiating $P_1(z)$ w.r.t. 'z' with the help of formula (21.7)′ we get

$$\frac{dP_1(z)}{dz} = \int_{x_1}^{z} (0 - c_2)\, f(x)\, dx + \left[\{(c_1 + c_2)\, x - c_2 z\} f(x) \frac{dx}{dz} \right]_{x_1}^{z}$$

$$= -c_2 \int_{x_1}^{z} f(x)\, dx + [(c_1 + c_2)\, z - c_2 z] f(z) = -c_2 \int_{x_1}^{z} f(x)\, dx + c_1 z f(z). \left(\text{since } \frac{dx_1}{dz} = 0 \right)$$

Similarly, $\frac{dP_2(z)}{dz} = \int_{z}^{x_2} c_1 f(x)\, dx + \left[c_1 z f(x) \frac{dx}{dz} \right]_{z}^{x_2} = c_1 \int_{z}^{x_2} f(x)\, dx - c_1 z f(z).$

Now, substituting the values of $dP_1(z)/dz$ and $dP_2(z)/dz$ in (21.10), we get

$$\frac{dP}{dz} = \left[-c_2 \int_{x_1}^{z} f(x)\, dx + c_1 z f(z) \right] + \left[c_1 \int_{z}^{x_2} f(x)\, dx - c_1 z f(z) \right] = 0$$

or $$-c_2 \int_{x_1}^{z} f(x)\, dx + c_1 \int_{z}^{x_2} f(x)\, dx = 0 \text{ or } -c_2 \int_{x_1}^{z} f(x)\, dx + c_1 \left[\int_{x_1}^{x_2} f(x)\, dx - \int_{x_1}^{z} f(x)\, dx \right] = 0$$

or $$-(c_1 + c_2) \int_{x_1}^{z} f(x)\, dx + c_1 = 0, \left(\text{since } \int_{x_1}^{x_2} f(x)\, dx = 1 \right)$$

or $$\int_{x_1}^{z} f(x)\, dx = \frac{c_1}{c_1 + c_2}. \qquad \text{...(21.11)}$$

Also, $d^2P/dz^2 = -(c_1 + c_2) f(z) < 0$ which is the sufficient condition of $P(z)$ being maximum.

In our problem, we are given that

$$c_1 = \text{Rs. } 5,\ c_2 = \text{Rs. } 1.20,\ x_1 = 2000,\ x_2 = 3000,\ f(x) = 1/(x_2 - x_1) = 1/1000.$$

Substituting these values in equation (21.11), we get

$$\int_{2000}^{z} \frac{1}{1000}\, dx = \frac{5.00}{5.00 + 1.20}$$

or $$\frac{1}{1000} [z - 2000] = 0.807 \text{ or } z = 2807 \text{ kg.}$$

Thus, optimal daily amount baked = 2807 kg. **Ans.**

Alternative. We may directly use the result (21.8) :

$$\int_{0}^{z} f(x)\, dx = \frac{C_2}{C_1 + C_2} \qquad \text{...(21.12)}$$

as obtained in **Model VI(b),** where

$C_1 = c_2$ = holding cost = Rs. 1.20, $f(x) = \frac{1}{1000}$, $C_2 = c_1$ = shortage cost = Rs. 5.00, $x_1 = 2000$, $x_2 = 3000$.

Substituting these values in (21.12), we get

$$\int_{2000}^{z} \frac{1}{1000}\, dx = \frac{5.00}{5.00 + 1.20} \qquad (\because \text{ lower limit of demand is 2000 instead of zero})$$

which gives rise to $z = 2807$ kg. as obtained above.

Example 6. *An ice-cream company sells one of its types of ice-cream by weight. If the product is not sold on the day it is prepared, it can be sold at a loss of 50 paise per kg. But there is an unlimited market for one day old ice-cream. On the other hand, the company makes a profit of Rs. 3.20 on every kg. of ice-cream sold on the day it is prepared. Past daily orders form a distribution with* $f(x) = 0.02 - 0.002x,\ 0 \le x \le 100$.

How many kg. of ice-cream should the company prepare every day ? **[Meerut (Maths.) 98, 97P]**

Solution. Here C_1 = Re. 0.50, C_2 = Rs. 3.20, $f(x) = 0.02 - 0.0002x$.

Directly using the result : $\int_{0}^{z} f(x)\, dx = \frac{C_2}{C_1 + C_2}$, we have

$$\int_0^z (0.02 - 0.0002x)\, dx = \frac{3.20}{0.50 + 3.20},$$

which gives $z = 63.2$ kg. **Ans.**

Q. 1. Discuss the problem of inventory control when the stochastic demand is uniform, production of commodity is instantaneous and lead time is negligible (discrete case)

2. Derive a single period probabilistic inventory model with instantaneous and continuous demand and no set up cost. **[Garhwal M.Sc. (Stat.) 92]**

21.3–5. Model VI (c) : Reorder Lead Time Prescribed

This model is similar to ***Model VI (a)*** with one important exception that the re-order lead time is to be taken into consideration. In other words, the time between placing an order and receiving the goods is significant here.

Formulation of the Cost Equation :

In this model, we are given that—

(i) n = the number of order cycle periods in the reorder lead time.

(ii) z_0 = the stock level at the end of the period preceding placing of the order.

(iii) $q_1, q_2, q_3, \dots, q_{n-1}$ = quantities already ordered and due to be received on the 1st, 2nd, 3rd, ..., and $(n-1)th$ days.

(iv) $f(x') = f(x_1 + x_2 + x_3 + \dots + x_n)$, where x' is the demand during the lead time.

Our problem is to find the value of quantity q_n in order to minimize the total expected cost of the nth order cycle time.

The minimization of the total expected cost from order cycle 1 to n is equivalent to minimizing the total cost for the nth period. Because, the total expected cost for the period from order cycle 1 to $n-1$ is already obtained since orders for quantities $q_1, q_2, q_3, \dots, q_{n-1}$ have already been placed.

The stock at the end of nth period can be obtained as below :

Obviously, we have

z_1 = [Stock level already present before placing the order + quantity already ordered which is due to be received in the Ist period]— demand during the first period.

or $z_1 = (z_0 + q_1) - x_1$.

Similarly,

$$z_2 = (z_1 + q_2) - x_2 = z_0 + (q_1 + q_2) - (x_1 + x_2)$$

$$z_3 = (z_2 + q_3) - x_3 = z_0 + (q_1 + q_2 + q_3) - (x_1 + x_2 + x_3)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$z_n = (z_{n-1} + q_n) - x_n = z_0 + (q_1 + q_2 + \dots + q_n) - (x_1 + x_2 + \dots + x_n)$$

$$= \left(z_0 + \sum_{i=1}^{n-1} q_i + q_n \right) - \sum_{i=1}^{n} x_i$$

or $$z_n = z' - x', \text{ say,}$$

where $$z' = z_0 + \sum_{i=1}^{n-1} q_i + q_n, \quad x' = \sum_{i=1}^{n} x_i .$$

Also, $$dz' = d\left(z_0 + \sum_{i=1}^{n-1} q_i + q_n \right)$$

or $$dz' = dq_n \quad \left(\text{since } z_0 \text{ and } \sum_{i=1}^{n-1} q_i \text{ are constant, their derivative is zero}\right)$$

Furthermore, from above, we observe that $z_n > 0$ when $z' > x'$, and $z_n < 0$ when $z' < x'$.

Now, simply substituting z' for z and x' for x in the cost eqn. (21.6) of ***Model VI (b),*** we get

$$C(z') = C_1 \int_0^{z'} (z' - x') f(x')\, dx' + C_2 \int_{z'}^{\infty} (x' - z') f(x')\, dx'.$$

Proceeding exactly as is ***Model VI (b),*** the optimum value of z_0' is that value which satisfies the equation

$$\int_0^{z'} f(x')\, dx' = \frac{C_2}{C_1 + C_2}.$$

After the optimum value of z' (*i.e.* z^*) is obtained, we can easily find the optimum value of q_n by using the relationship

$$q_n^* = z^* - \left(z_0 + \sum_{i=1}^{n-1} q_i \right)$$

Q. 1. Formulate and solve continuous probabilistic reorder point lot size model to determine optimal reorder point for a presented lot size. Lead time is finite. Shortages are allowed and fully backlogged.

2. Explain with examples the probabilistic models in inventory. **[Garhwal M.Sc. (Stat.) 93]**

Example 7. *A shop owner places orders daily for goods which will be delivered 7 days later (i.e., the reorder lead time is 7 days). On a certain day, the owner has 10 items in stock. Furthermore, on the 6 previous days, he has already placed orders, for the delivery of 2, 4, 1, 10, 11 and 5 items in that order, over each of the next 6 days. To facilitate computations, we shall assume* $C_1 = Re.\ 0.15$, $C_2 = Re.\ 0.95$ *and the distribution requirement over a 7-day period* (x') *is* $f(x') = 0.02 - 0.0002x'$.

How many items should be ordered for the 7th day hence ?

Solution. Substituting the values of C_1, C_2 and $f(x')$ in the result

$$\int_0^{z'} f(x')\, dx' = \frac{C_2}{C_1 + C_2},$$

we get $$\int_0^{z'} (0.02 - 0.0002x')\, dx' = \frac{0.95}{0.15 + 0.95} \quad \text{or} \quad \left[0.02\, x' - 0.0002 \left(\frac{x'^2}{2} \right) \right]_0^{z'} = 0.8636$$

or $$0.02\, z' - 0.0001\, z'^2 = 0.8636 \quad \text{or} \quad 0.0001 z'^2 - 0.02\, z' + 0.8636 = 0$$

or $$z'^2 - 200\, z' + 8600 = 0.$$

Solving this quadratic equation we (approximately) get $z' = 63$ or 136 items.

Since $z' = z_0 + \sum_{i=1}^{6} q_i + q_7$, we have $63 = 10 + (2 + 4 + 1 + 10 + 11 + 5) + q_7$,

or $$q_7 = 63 - 10 - 33 = 20 \text{ items.}$$

Thus, the optimum order quantity is 20 items.

EXAMINATION PROBLEMS (On Model VI)

1. A T.V. dealer finds that cost of holding a television in stock for a week is Rs. 20, customers who cannot obtain new television immediately tend to go to other dealers; and he estimates that for every customer who does not get immediate delivery he loses on an average Rs. 200. For on particular model of television the probabilities for a demand of 0, 1, 2, 3, 4 and 5 televisions in a week are 0.05, 010, 0.20, 0.30, 0.20 and 0.15 respectively. How many televisions per week should the dealer order ? (Assume that there is no time lag between ordering and delivery).
   [**Hint.** C_1 = Rs. 20, C_2 = Rs. 200. Use the formula (3.4) of **Model VI(a)** and get $0.85 < 0.921 < 1.00$.]
   [**Ans.** The order for televisions per week should be 4.]
2. A company uses to order a new machine after a certain fixed time. It is observed that one of the parts of the machine is very expensive if it is ordered without machine. The cost of spare part when ordered with the machine is Rs. 500.00 The cost of down time of the machine and the cost of arranging the new parts is Rs. 10,000. From the past record it is observed that spare part is required with the probabilities mentioned below :

Demand (r) :	0	1	2	3	4	5	6 or more
Probablity $P(r)$:	0.90	0.05	0.02	0.01	0.01	0.01	0.00

Find the optimal number of spare parts which should be ordered with the spare parts with order of the machine.
[**Hint.** $C_1 = 500$, $C_2 = 10{,}000$. Use the formula (3.4) of **Model VI(a),** and get $0.95 < 0.952 < 0.97$.]
[**Ans.** 2 spare parts must be ordered with machine.]

3. The probability distribution of monthly sales of a certain item is as follows :

Monthly sales :	0	1	2	3	4	5	6
Probability :	0.01	0.06	0.25	0.35	0.20	0.03	0.10

The cost of carrying inventory is Rs. 30 per unit per month and the cost of unit shortage is Rs. 70 per month. Determine the optimum stock level which minimizes the total expected cost. **[Meerut (M.Sc.) 96]**

[**Hint.** C_1 = Rs. 30.00 per unit per month, C_2 = Rs. 70 per unit-per month. Using the formula (3.4), obtain

$$0.67 < \left[\frac{C_2}{C_1 + C_2} = 0.7\right] < 0.87$$

[**Ans.** Monthly sales of given item is 3.]

4. Two products are stocked by a company. The company has limited space and cannot store more than 40 units. The demand distribution for the products are as follows :

Demand (z) :	0	10	20	30	40
Prob. for Ist product :	0.10	0.20	0.35	0.25	0.10
Prob. for IInd product :	0.05	0.20	0.40	0.20	0.15

The inventory carrying costs are Rs 5 and Rs. 10 per unit of the ending inventories for the first and second product, respectively. The shortage costs are Rs. 20 and Rs. 50 per unit of the ending shortages for the first and second product, respectively. Find out the economic order quantities for both the products.

[**Hint.** For the two products, $C_2/(C_1 + C_2)$ lies between (0.65 and 0.90) and (0.65 and 0.85) respectively. Hence for both the products optimum order quantity is **20 units. Ans.**]

5. A baking company sells one of its types of cakes by weight. It makes a profit of 95 paise a pound on every pound of cake sold on the day it is baked. It disposes of all cake not sold on the date it is baked at a loss of 15 paise a pound. If demand is known to be rectangular between 3000 and 4000 pounds, determine the optimum amount baked.

[Garhwal M.Sc. (Math.) 96]

[**Hint.** Proceed exactly as solved example 5. **Ans.** 3,864]

6. A fish stall sells a variety of fish at the rate of Rs. 5.0 per kg on the day of the catch. If the stall fails to sell the catch on the same day, it pays for shortage at the rate of Re. 0.30 per kg. and the price fetched is Rs. 4.50 per kg on the next day. Past record shows that there is an unlimited demand for fish one day old. The problem is to ascertain how much fish should be procured every day, so that the total expected cost is minimum. It has been found from the past records that daily demand follows as exponential distribution with $f(x) = 0.02\, e^{-0.02x}$, $0 \le x \le \infty$.

[**Hint.** C_1 = Re. 0.30 + Rs. (5.00 – 4.50) = Re. 0.80, C_2 = Rs. 5.00. Using the formula

$$\int_0^z f(x)\, dx = C_2/(C_1 + C_2)$$

get $e^{-0.02z} = 1.862$. By trial and error method find **z = 100 kg. Ans.**]

7. Let the probability density of a certain item during a day be

$$f(x) = \begin{cases} 0.1, & 0 \le x \le 10 \\ 0, & x > 10. \end{cases}$$

The demand is assumed to occur with a uniform pattern during the whole day. Let the unit carrying cost of the item in inventory be Re. 0.50 per day and unit shortage cost be Rs. 4.5 per day. If Re. 0.50 be the purchasing cost per unit, determine the optimal order level of the inventory.

[**Hint.** C_1 = Re. 0.50, C_2 = Rs. 4.5, C = unit purchasing cost = Re. 0.50. Use the formula

$$\int_0^z f(x)\, dx = \frac{C_2 - C_1}{C_1 + C_2} \quad \text{and find } z = 8 \text{ units. } \textbf{Ans.}]$$

8. A newspaper boy sells papers to his customers. He makes a new profit of 20 paise per paper sold. He disposes of all unsold paper at a loss of 10 paise per paper. If the distribution of demand is rectangular between 400 and 500, determine the optimal number of newspapers so as to maximize his new profit. **[Delhi M.Sc. (OR.) 90]**

9. A shopkeeper has to decide how much quantity of bread he should stock every week. The quantity of bread demand in any week is assumed to be continuous randem variable with a given probability distribution function $f(x)$. Let A = Rs. 8 be the unit cost of purchasing the bread, B = Rs. 20 unit sale price, C = Rs. 2 refund on unit sale bread and D = Rs. 5 is the unit penalty cost. Find the optimum quantity of bread to be stocked. **[JNTU (B. Tech.) 2003]**

21.4. UNIFORM DEMAND, NO SET-UP COST MODEL

Under this probabilistic model, the type of problem is similar to that considered under the *Model VI*, except that withdrawals from stock are continuous (rather than instantaneous) and rate of withdrawals is assumed to be virtually constant. Here demand distributions are dependent over time.

21.4–1. Model VII (a) : Continuous Version.

To find the optimum order level so as to minimize the total expected cost, where

(i) t is the scheduling period which is a prescribed constant;
(ii) z is the stock level to which the stock is raised at the end of every period t;
(iii) f (x) is the probability density function for demand x which is known;
(iv) C_1 *is the carrying cost per quantity unit per unit time;*
(v) C_2 *is the shortage cost per quantity unit per unit time;*
(vi) the production is instantaneous; (vii) lead time is zero; and
(viii) the demand rate in a period t is constant.

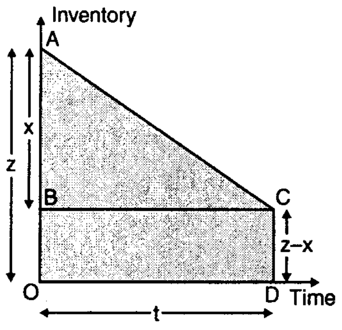


Fig. 21.4
Average holding inventory = $z - ½\, x$
Average shortage inventory = 0

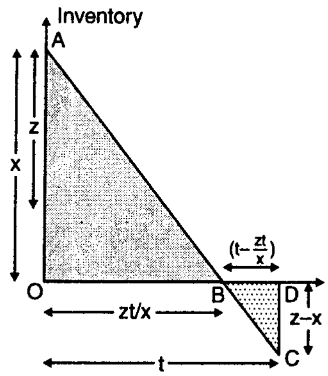


Fig. 21.5
Average holding inventory = $z^2/2x$
Average shortage inventory = $(x-z)^2/2x$

Solution. In this case, demand occurs uniformly rather than instantaneously during period t, as shown in *Fig.* 21.4 and 21.5. There are two possibilities :

(i) $x \le z$ **:** In this situation there is no shortage and only carrying cost is involved. Holding cost, as seen from *Fig.* 21.4 is = C_1. [Inventory Area *AODC*]
$= C_1 \,.\, ½\, [CD + AO] \times BC$ (by formula for area of trapezium) $\quad = C_1 \,.\, ½\, [2z - x]\, t.$

(ii) $x > z$ **:** In this situation the shortage cost is also involved. We see in *Fig.* 21.5 that the area of inventory

$$\Delta AOB = ½\, z.\ \frac{zt}{x} = \frac{z^2 t}{2x}, \left[\text{from similar } \Delta AEC \text{ and } \Delta AOB\text{, we get } \frac{AO}{AE} = \frac{OB}{EC} \text{ or } \frac{z}{x} = \frac{OB}{t},\ \therefore\ OB = \frac{zt}{x}\right].$$

and in *Fig.* 21.5. shortage area $\Delta BDC = ½\, BD \times CD = ½\left(t - \frac{zt}{x}\right)(x - z) = \frac{1}{2x}(x - z)^2\, t$

Total average cost becomes $= C_1 \,.\, \frac{z^2}{2x} + C_2 \,.\, \frac{1}{2x}(x - z)^2.$

Thus, total expected cost is given by the ***cost equation*** :

$$C(z) = \underbrace{\int_0^z \frac{C_1}{2}(2z - x) f(x)\, dx}_{(x \le z)} + \underbrace{\int_z^\infty \left[\frac{C_1 z^2}{2x} + \frac{C_2 (x - z)^2}{2x}\right] f(x)\, dx}_{(x > z)}. \qquad \text{...(21.13)}$$

Differentiating (21.13) with respect to z, we get

$$\frac{dC}{dz} = C_1 \int_0^z (1 - 0) f(x)\, dx + C_1 \left[\left(z - \frac{x}{2}\right) f(x) \frac{dx}{dz}\right]_0^z + \int_z^\infty \left[\frac{C_1}{2x}(2z) - \frac{C_2}{2x} 2(x - z)\right] f(x)\, dx$$
$$+ \left[\left(\frac{C_1 z^2}{2x} + \frac{C_2 (x - z)^2}{2x}\right) f(x) \frac{dx}{dz}\right]_z^\infty$$

$$= C_1 \int_0^z f(x)\, dx + C_1 \frac{z}{2} f(z) + \int_z^\infty \left[(C_1 + C_2) \frac{z}{x} - C_2 \right] f(x)\, dx + \left[0 - \left(\frac{C_1 z^2}{2z} + 0 \right) f(z) \right]$$

$$= C_1 \int_0^z f(x)\, dx + \frac{C_1 z}{2} f(z) + \int_z^\infty \left[(C_1 + C_2) \frac{z}{x} - C_2 \right] f(x)\, dx - \frac{C_1 z f(z)}{2}$$

$$= C_1 \int_0^z f(x)\, dx + \int_z^\infty (C_1 + C_2) \frac{z f(x)\, dx}{x} - C_2 \int_z^\infty f(x)\, dx$$

$$= C_1 \int_0^z f(x)\, dx + \int_z^\infty (C_1 + C_2) \frac{z f(x)\, dx}{x} - C_2 \left[\int_0^\infty f(x)\, dx - \int_0^z f(x)\, dx \right].$$

$$\therefore \quad \frac{dC}{dz} = (C_1 + C_2) \int_0^z f(x)\, dx + (C_1 + C_2) \int_z^\infty \frac{z f(x)}{x}\, dx - C_2 . \qquad \left(\because \int_0^\infty f(x)\, dx = 1 \right) \quad ...(21.14)$$

For minimum cost, $dC/dz = 0$, therefore,

$$(C_1 + C_2) \int_0^z f(x)\, dx + (C_1 + C_2) \int_z^\infty \frac{z}{x} f(x)\, dx - C_2 = 0$$

or

$$\int_0^z f(x)\, dx + z \int_z^\infty \frac{f(x)\, dx}{x} = \frac{C_2}{C_1 + C_2}. \qquad ...(21.15)$$

This equation gives the optimum value of z for minimum expected cost, provided d^2C/dz^2 is positive.

$$\therefore \quad \frac{d^2C}{dz^2} = (C_1 + C_2) \int_z^\infty \frac{f(x)}{x}\, dx + \left[(C_1 + C_2) \frac{z}{x} f(x) \frac{dx}{dz} \right]_z^\infty + (C_1 + C_2) \Big[f(x) \Big]_0^z$$

$$= (C_1 + C_2) \int_z^\infty \frac{f(x)}{x}\, dx + (C_1 + C_2)\, [0 - f(z)] + (C_1 + C_2) f(z) = \text{ a positive quantity.}$$

Hence C is minimum for optimum value of z given by (21.15).

Q. 1. Find the optimal quantity z in a continuous simple stochastic model for a time dependent case. Shortages are allowed and backlogged fully, setup cost per period is constant.

2. Discuss a probabilistic reorder point lot size inventory model to determine the optimal reorder point for a prescribed lot size. Demand is random and continuous with given probability density function. Shortages are not allowed. Lead time is zero. Usual notations may be used. **[Delhi M.Sc. (OR) 92]**

3. Formulate and solve a continuous stochastic model for a single producer. The storage and shortage costs depend on unit and time. The set up cost is negligible. Using the transformation

$$\phi(y) = \begin{cases} 0 & \text{for } y = 0 \\ \int_{x=y}^\infty \frac{f(x)}{x}\, dx & \text{for } y > o, \end{cases}$$

Show that the above model is equivalent to another model with storage and shortage costs independent of time. **[Delhi. M.Sc. (OR) 92, 90]**

Example 8. *Let the probability density of demand of a certain item during a week be*

$$f(x) = \begin{cases} 0.1 & 0 \le x \le 10 \\ 0, & otherwise \end{cases}$$

This demand is assumed to occur with a uniform pattern over the week. Let the unit carrying cost of the item in inventory be Rs. 2.00 per week and unit shortage cost be Rs. 8.00 per week. How will you determine the optimal order level of the inventory ? **[Meerut (Stat.) 95]**

Solution. Since $f(x) = 0.1$, $0 \le x \le 10$, $C_1 =$ Rs. 2.00, $C_2 =$ Rs. 8.00, then using the result (21.15), we get

$$\int_0^z (0.1)\, dx + z \int_z^{10} \frac{0.1}{x}\, dx = \frac{8}{10},$$

or

$$0.1\,(z - z \log z + 2.3 z) = 0.8 \quad \text{or} \quad 3.3 z - z \log z - 8 = 0.$$

The solution of this equation is obtained by trail and error method which is given by $z = 4.5$.

Note. The students are also advised to solve this problem for instantaneous demand then notice the difference between the two results.

21.4-2. Model VII (b) : Discrete Version

The cost equation for this model is similar to that derived for *Model VII(a)*. We simply replace $f(x)\, dx$ by probability $P(x)$, and the integration by summation. Then, the cost equation becomes

$$C(z) = \sum_{x=0}^{z} C_1\left(z - \frac{x}{2}\right)P(x) + \sum_{x=z+1}^{\infty}\left(\frac{C_1 z^2}{2x} + \frac{C_2 (x-z)^2}{2x}\right)P(x). \quad ..(21.16)$$

For minimum value of $C(z)$, we require that z should satisfy the relationship

$$\Delta C(z-1) < 0 < \Delta C(z). \quad ...(21.17)$$

But, we can difference under the summation sign if, for $x = z + 1$

$$C_1 \sum_{x=0}^{z}\left(z+1-\frac{x}{2}\right)P(x) \equiv C_1 \sum_{x=z+1}^{\infty}\frac{(z+1)^2}{2x}P(x) + C_2 \sum_{x=z+1}^{\infty}\frac{[x-(z+1)]^2}{2x}P(x) \quad ...(21.18)$$

We observe that both sides of (21.18) are identically equal for $x = z + 1$. So we may find the first difference of equation (21.16) under the summation sign.

By difference calculus, we know that $\Delta C(z) = C(z+1) - C(z)$.

Hence, applying this formula to each term of (21.16), we get

$$\Delta C(z) = C_1 \sum_{x=0}^{z}\left\{\left((z+1)-\frac{x}{2}\right)-\left(z-\frac{x}{2}\right)\right\}P(x) + C_1 \sum_{x=z+1}^{\infty}\left[\frac{(z+1)^2}{2x} - \frac{z^2}{2x}\right]P(x)$$

$$+ C_2 \sum_{x=z+1}^{\infty}\left[\frac{(x-z-1)^2}{2x} - \frac{(x-z)^2}{2x}\right]P(x)$$

$$= C_1 \sum_{x=0}^{z} P(x) + C_1 \sum_{x=z+1}^{\infty}\frac{(2z+1)}{2x}P(x) - C_2 \sum_{x=z+1}^{\infty}\frac{(2x-2z-1)}{2x}P(x)$$

$$= C_1 \sum_{x=0}^{z} P(x) + C_1 (z + 1/2) \sum_{x=z+1}^{\infty}\frac{P(x)}{x} + C_2 (z + 1/2) \sum_{x=z+1}^{\infty}\frac{P(x)}{x} - C_2 \sum_{x=z+1}^{\infty} P(x)$$

$$= C_1 \sum_{x=0}^{z} P(x) + (C_1 + C_2)(z + 1/2) \sum_{x=z+1}^{\infty}\frac{P(x)}{x} - C_2\left[\sum_{x=0}^{\infty} P(x) - \sum_{x=0}^{z} P(x)\right]$$

$$= (C_1 + C_2) \sum_{x=0}^{z} P(x) + (C_1 + C_2)(z + 1/2) \sum_{x=z+1}^{\infty}\frac{P(x)}{x} - C_2 \qquad \left\{\text{since } \sum_{x=0}^{\infty} P(x) = 1\right\}$$

But $\Delta C(z) > 0$ for minimum $C(z)$, therefore

$$\sum_{x=0}^{z} P(x) + (z + 1/2) \sum_{x=z+1}^{\infty}\frac{P(x)}{x} > \frac{C_2}{C_1 + C_2} \quad ...(21.19)$$

The value of z satisfying the relation (21.19) will be the optimum order level.

By using the condition $\Delta C(z-1) < 0$ for minimum $C(z)$, we can also obtain the relation

$$\sum_{x=0}^{z-1} P(x) + (z - 1/2) \sum_{x=z}^{\infty}\frac{P(x)}{x} < \frac{C_2}{C_1 + C_2} \quad ...(21.20)$$

Combine (21.19) and (21.20) in the form

$$\sum_{x=0}^{z-1} P(x) + (z - 1/2) \sum_{x=z}^{\infty}\frac{P(x)}{x} < \frac{C_2}{C_1 + C_2} < \sum_{x=0}^{z} P(x) + (z + 1/2) \sum_{x=z+1}^{\infty}\frac{P(x)}{x} \quad ...(21.21)$$

By using the relationship (21.21), we can find the range of optimum value of z.

Q. 1. Derive the rule that gives optimum order quantity for a single period stochastic inventory system for which holding cost and shortage cost are proportional to time and quantity. Assume that the demand is discrete.
[Delhi MA/M.Sc. (Stat.) 95]

2. (a) What considerations are inherent in inventory management ? Discuss in detail the impact of the patterns of demand and lead time in finding the optimal inventory policy.

(b) Derive the economic lot size formula when shortages are not permissible.

(c) Given the holding cost c_1 per unit, the shortage cost c_2 per unit and probabilities $P(r \leq s)$, r denoting the number of spare parts required and s the inventory level, obtain analytically the optimum inventory level that minimizes the total expected cost. How would you ascertain $P(r \leq s)$ in practical applications ?

21.4-3. Illustrative Example—To Estimate the Cost of Shortage.

Example 9. *The probability distribution of monthly sales of a certain item is as follows :*

Monthly sales :	0	1	2	3	4	5	6
Probability :	0.02	0.05	0.30	0.27	0.20	0.10	0.06

The cost of carrying inventory is Rs. 10.00 per unit per month. The current policy is to maintain a stock of four items at the beginning of each month. Assuming that the cost of shortage is proportional to both time and quantity short, obtain the imputed cost of a shortage of one item for one time unit.

(Because the problem is stated in discrete units, the answer will consist of a range of values for the imputed cost). **[Meerut (Maths.) Jan. 98 BP, 93P]**

Solution. In this problem, we are given that

(1) optimum stock, $z = 4$ items,

(2) carrying cost C_1 = Rs. 10.00 per item per month,

(3) the probability $p(x)$ for sale x in each month is as follows :

$p(0)$	$p(1)$	$p(2)$	$p(3)$	$p(4)$	$p(5)$	$p(6)$
0.02	0.05	0.30	0.27	0.20	0.10	0.06,

(4) the shortage cost C_2 is to be determined,

(5) the range of monthly sales x is given from 0 to 6 times in discrete units (not from 0 to ∞ here),

We have proved the relationship (21.21) :

$$\sum_{x=0}^{z-1} p(x) + (z - \tfrac{1}{2}) \sum_{x=z}^{\infty} \frac{p(x)}{x} < \frac{C_2}{C_1 + C_2} < \sum_{x=0}^{z} p(x) + (z + \tfrac{1}{2}) \sum_{x=z+1}^{\infty} \frac{p(x)}{x}$$

Now least value of C_2 can be determined by letting

$$\frac{C_2}{C_1 + C_2} = \sum_{x=0}^{z-1} p(x) + (z - \tfrac{1}{2}) \sum_{x=z}^{\infty} \frac{p(x)}{x}$$

Therefore, substituting the given values, we get

$$\frac{C_2}{10 + C_2} = \sum_{x=0}^{3} p(x) + (4 - \tfrac{1}{2}) \sum_{x=4}^{6} \frac{p(x)}{x}$$

$$= [p(0) + p(1) + p(2) + p(3)] + \tfrac{7}{2}\left[\frac{p(4)}{4} + \frac{p(5)}{5} + \frac{p(6)}{6}\right]$$

$$= (0.02 + 0.05 + 0.30 + 0.27) + \tfrac{7}{2}\left[\frac{0.20}{4} + \frac{0.10}{5} + \frac{0.06}{6}\right] = 0.92.$$

$\therefore$ Least value of $C_2 = 9.2/.08$ = Rs. 115.

Similarly, the greastest value of C_2 can be determined by letting

$$\frac{C_2}{C_1 + C_2} = \sum_{x=0}^{z} p(x) + (z + \tfrac{1}{2}) \sum_{x=z+1}^{\infty} \frac{p(x)}{x}$$

Substituting the given values, we get

$$\frac{C_2}{10 + C_2} = \sum_{x=0}^{4} p(x) + (4 + \tfrac{1}{2}) \sum_{x=5}^{6} \frac{p(x)}{x}$$

$$= 0.84 + \tfrac{9}{2} \times 0.03 = 0.975.$$

$\therefore$ Greatest value of $C_2 = 9.75/.025$ = Rs. 390.

Hence, the required range of values for the imputed cost C_2 can be written as Rs. 115 < C_2 < Rs. 390.

EXAMINATION PROBLEM

1. A cycle dealer finds that the cost of holding a cycle in stock for a week is Rs. 10. Customers who cannot obtain new cycles immediately tend to go to other dealers, and he estimates that for every customer who cannot get immediate delivery he loses an average of Rs. 80. For one particular model of cycle the probabilities of a demand of 0, 1, 2, 3, 4, and 5 cycles in a week are 0.05, 0.10, 0.20, 0.30, 0.20 and 0.15 respectively. How many cycles should the dealer keep in stock per week ? (Assume that there is no time lag between ordering and delivery). **[Delhi (M.B.A.) Dec. 94]**
2. Demand for a particular product is probabilistic and is as follows.

x	:	0	1	2	3	4
$p(x)$	:	0.01	0.20	0.39	0.20	0.20

The cost of producing one unit is Rs. 5,000.00. I_c = Rs.500 per unit per year, π = Rs. 1,500 per unit per year. Find optimal production level. **[Delhi M.Sc. (OR.) 92]**

21.5. PROBABILISTIC ORDER-LEVEL SYSTEM WITH CONSTANT LEAD TIME

This model is similar to the previous one with one important exception that the reorder lead time is significant. So, we must take into account the time between placing an order and delivery of goods ordered. Both the versions of this model : *Discrete* and *continuous*, are discussed in the following two subsections.

21.5–1. Model VIII (a) : Discrete Units

In this model, we have to find the optimum order level z so as to minimize the total expected cost, where

(i) t_p is the prescribed interval between orders,

(ii) x is the demand during the period t_p with probability $F(x)$,

(iii) y is the demand during the lead time L = T with probability $G(y)$,

(iv) C_1 is the holding cost per item per unit time,

(v) C_2 is the shortage cost per item per unit time.

Solution. First, we should remember that the replenishment cost C_3 is not involved in this model, since t_p is a prescribed constant. It is also important to note that the amount in stock during the specific period depend on the order level z, the demand y during the lead time, and the demand x during t_p. There are three possible situations depending on the relative values of x, y and z. The graphical representation of these three possible situations is given in *Figs.* 21.6, 21.7 *and* 21.8.

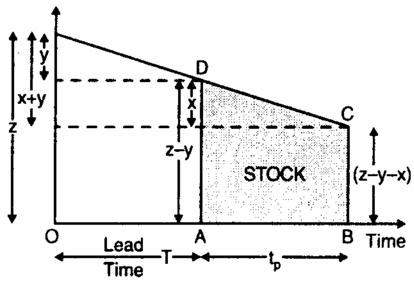


Fig. 21.6 Situation *A* : $0 \le y \le z$; $0 < x \le z - y$.

It is quite obvious that the present decision will affect the situation during the period t_p. Period t_p starts when the lead time T has elapsed. Thus, our aim is to minimize the expected costs during t_p.

First, we shall find the total (actual) costs in three different situations : *A*, *B*, and *C* that may arise here.

(1) Situation A. In this situation, the period t_p starts with the stock $z - y$, and ends with the remaining stock $[(z - y) - x]$ so that the total (actual) cost becomes

$$= C_1 \times [\text{Inventory area } ABCD]$$

$$= C_1 \cdot \tfrac{1}{2}\,[(z-y) + (z-y-x)]\, t_p$$

$$= C_1\,(z - y - x/2)\, t_p$$

(2) Situation B. In this situation, the period t_p starts with a stock $(z - y)$ and ends with shortage $(x + y - z)$. As we have already seen in the *Model II (a)* **[*Chapter* 20]**, the area representing inventory cost is

$$= \tfrac{1}{2}\,(z-y)\left(\frac{z-y}{x}\right)t_p = \frac{1}{2x}\,(z-y)^2\, t_p\,,$$

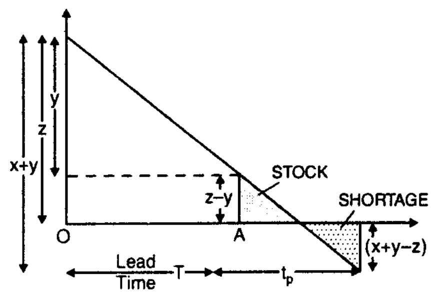


Fig. 21.7 situation B : $0 \le y \le z$; $x > z - y$

and the area representing shortage cost is

$$= \tfrac{1}{2}(x+y-z)\left[t_p - \frac{z-y}{x}t_p\right] = \frac{1}{2x}[x+y-z]^2 t_p.$$

Thus, we obtain the total cost in this situation

$$= C_1 \frac{1}{2x}(z-y)^2 t_p + C_2 \frac{1}{2x}[x+y-z]^2 t_p.$$

(3) Situation C. In this situation, the period t_p starts with a shortage $(y - z)$ and ends with a shortage $(x + y - z)$, so that the shortage cost

$$= C_2 \times [\text{shortage area } ABCD]$$
$$= C_2 \times \tfrac{1}{2}[y - z + (x + y - z)]\, t_p$$
$$= C_2\,(x/2 + y - z)\, t_p.$$

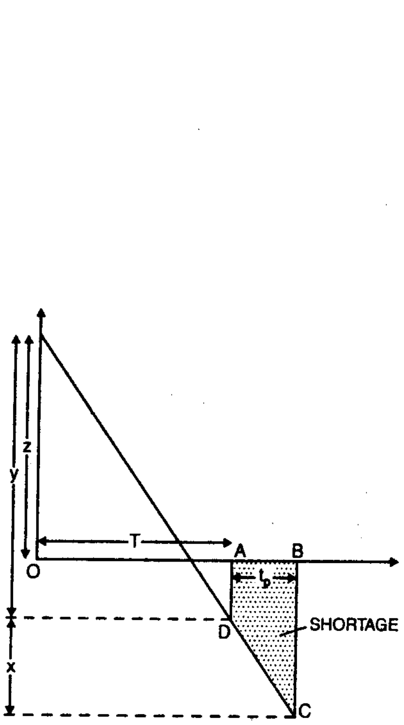


Fig. 21.8 Situation $C: y > z;\ x \geq 0$.

Now, the total expected cost $C(z)$ per unit time (excluding C_3) is obtained by multiplying the cost associated with each situation by the joint probability of demands for y items during the time T, and summing over the appropriate range of x and y. Hence, the cost equation for this model becomes

$$C(z) = \sum_{y=0}^{z}\sum_{x=0}^{z-y} C_1(z-y-x/2)F(x)G(y) + \sum_{y=0}^{z}\sum_{x=z-y+1}^{\infty}\frac{1}{2x}[C_1(z-y)^2 + C_2(x+y-z)^2]F(x)G(y)$$

$$+ \sum_{y=z+1}^{\infty}\sum_{x=0}^{\infty} C_2(x/2+y-z)F(x)G(y). \qquad ...(21.22)$$

Now, for minimum $C(z)$, we require $\Delta C(z) > 0 > \Delta C(z-1)$.

Therefore, in order to find $\Delta C(z)$, we proceed as explained in *Case 2* of **sec. 2.29-2**, page 43.

The summation is taken over the first quadrant of xy-plane, divided into three parts as shown in *Fig.* 21.9.

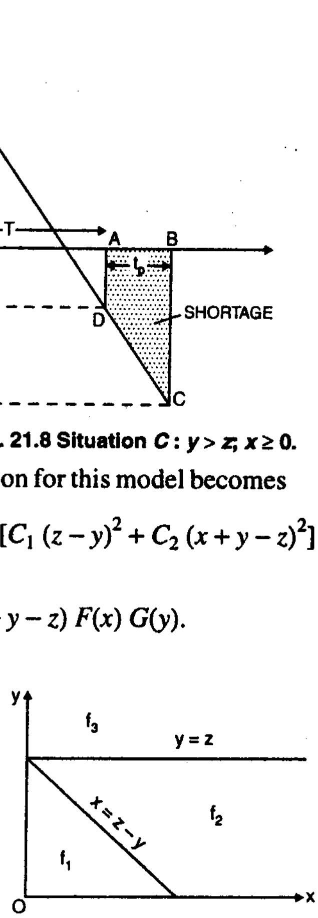


Fig. 21.9

Here

$$f_1(x, y, z) = C_1(z - y - x/2)F(x)G(y),$$

$$f_2(x, y, z) = \frac{1}{2x}[C_1(z-y)^2 + C_2(x+y-z)^2]F(x)G(y)$$

$$f_3(x, y, z) = C_2(x/2 + y - z)F(x)G(y),$$

$$b(z) = z,\ c(y, z) = z - y,\ d(y, z) = +\infty$$

$$\therefore\ \Delta C(z) = \sum_{y=0}^{z}\sum_{x=0}^{z-y} C_1 F(x)G(y) - \sum_{y=z+1}^{\infty}\sum_{x=0}^{\infty} C_2 F(x)G(y)$$

$$+ \sum_{y=0}^{z}\sum_{x=z-y+1}^{\infty}\frac{1}{x}[(C_1+C_2)(z-y+\tfrac{1}{2}) - C_2]F(x)G(y)$$

$$+ \sum_{y=0}^{z}\sum_{x=z-y+1}^{\infty}\left[C_1(z+1-y-x/2) - \frac{C_1}{2x}(z+1-y)^2 + \frac{C_2}{2x}(x+y-z-1)^2\right]F(x)G(y)$$

$$+ \sum_{y=z+1}^{z+1}\left[\sum_{x=0}^{z-y+1} C_1(z+1-y-x/2)F(x) + \sum_{x=z-y+2}^{\infty}\left\{\frac{C_1}{2x}(z+1-y)^2 + \frac{C_2}{2x}(x+y-z-1)^2\right\}F(x)\right.$$

$$\left. - \sum_{x=0}^{\infty} C_2(x/2+y-z-1)F(x)\right]G(y). \qquad ...(21.23)$$

We observe that all except first three terms reduce to single summations, and in fact, all except the first three terms vanish.

Thus,

$$\Delta C(z) = C_1 \sum_{y=0}^{z} \sum_{x=0}^{z-y} F(x)\, G(y) + \sum_{y=0}^{z} \sum_{x=z-y+1}^{\infty} \frac{1}{2x}\left[(C_1 + C_2)(2z - 2y + 1) - 2C_2 x \right] F(x)\, G(y)$$

$$- C_2 \sum_{y=z+1}^{\infty} \sum_{x=0}^{\infty} F(x)\, G(y) \qquad \text{...(21.24)}$$

$$= C_1 \sum_{y=0}^{z} \sum_{x=0}^{z-y} F(x)\, G(y) + (C_1 + C_2) \sum_{y=0}^{z} \sum_{x=z-y+1}^{\infty} \frac{1}{x}(z - y + ½)\, F(x)\, G(y)$$

$$- C_2 \sum_{y=0}^{z} G(y) \left[\sum_{x=0}^{\infty} F(x) - \sum_{x=0}^{z-y} F(x) \right] - C_2 \left[\sum_{y=0}^{\infty} \sum_{x=0}^{\infty} F(x)\, G(y) - \sum_{y=0}^{z} \sum_{x=0}^{\infty} F(x)\, G(y) \right]$$

Now, using the relations : $\sum_{x=0}^{\infty} F(x) = 1, \quad \sum_{y=0}^{\infty} G(y) = 1$, we have

$$\Delta C(z) = C_1 \sum_{y=0}^{z} \sum_{x=0}^{z-y} F(x)\, G(y) + (C_1 + C_2) \sum_{y=0}^{z} \sum_{x=z-y+1}^{\infty} \frac{1}{x}(z - y + ½)\, F(x)\, G(y)$$

$$- C_2 \sum_{y=0}^{z} G(y) \left[1 - \sum_{x=0}^{z-y} F(x) \right] - C_2 \left[1 - \sum_{y=0}^{z} G(y) \right]$$

$$= (C_1 + C_2) \left[\sum_{y=0}^{z} \sum_{x=0}^{z-y} F(x)\, G(y) + \sum_{y=0}^{z} \sum_{x=z-y+1}^{\infty} \frac{1}{x}(z - y + ½)\, F(x)\, G(y) \right] - C_2 . \qquad \text{...(21.25)}$$

Thus, we can find the optimum value of z for which $\Delta C(z) \geq 0$, *i.e.*

$$\sum_{y=0}^{z} G(y) \left[\sum_{x=0}^{z-y} F(x) + (z - y + ½) \sum_{x=z-y+1}^{\infty} \frac{F(x)}{x} \right] \geq \frac{C_2}{C_1 + C_2} . \qquad \text{..(21.26)}$$

21.5-2. Model VIII (b) : Continuous Units

The cost equation for this model is similar to that of discrete version. Only $F(x)$ and $G(y)$ are replaced by probability functions $f(x)\,dx$ and $g(y)\,dy$ respectively, and double summation ($\Sigma\Sigma$) is replaced by double integral ($\int\int$). Then the cost equation becomes :

$$C(z) = \int_0^z g(y)\, dy \int_0^{z-y} C_1 (z - y - x/2) f(x)\, dx + \int_0^z g(y)\, dy \int_{z-y}^{\infty} \left[C_1 \frac{(z-y)^2}{2x} + C_2 \frac{(x - z + y)^2}{2x} \right] f(x)\, dx$$

$$+ \int_z^{\infty} g(y)\, dy \int_0^{\infty} C_2 (y - z + x/2) f(x)\, dx. \qquad \text{...(21.27)}$$

Now, differentiating (21.27) by using the formula for differentiation of integrals, and simplifying we get the following result analogous to (21.25),

$$\frac{dC}{dz} = (C_1 + C_2) \int_0^z g(y)\, dy \left[\int_0^{z-y} f(x)\, dx + (z - y) \int_{z-y}^{\infty} \frac{f(x)}{x}\, dx \right] - C_2 = 0. \qquad \text{...(21.28)}$$

Again, differentiating (21.28), we get

$$\frac{d^2C}{dz^2} = (C_1 + C_2) \int_0^z g(y)\, dy \int_{z-y}^{\infty} \frac{f(x)}{x}\, dx > 0. \qquad \text{...(21.29)}$$

Thus optimum order level is determined by

$$\int_0^z g(y)\, dy \left[\int_0^{z-y} f(x)\, dx + (z - y) \int_{z-y}^{\infty} \frac{f(x)}{x}\, dx \right] = \frac{C_2}{C_1 + C_2} . \qquad \text{...(21.30)}$$

Q. 1. Determine the optimum value of the order quantity for a simple stochastic model when shortages are allowed and backlogged. Inventory carrying cost and shortage cost are dependent on units and time both. Procurement lead time is a random variable.

Example 10. *An airline runs a school for air hostesses each month. It takes two months to assemble a group of girls and to train them. Past records of turnover in hostesses show that the probability of requiring x new trained hostesses in any one month is g(x)* [$x = 0, 1, 2, \ldots$], *and the probability of requiring y new hostesses in any two-month period is h(y). In the event that a trained hostess is not required for flying duties, the airline still has to pay her salary at the rate of C_1 per month. If insufficient hostesses are available, there is a cost of C_2 per girl short per month. Show how to determine decision rules for the size of classes.*

Solution. This problem is based on *Model VIII (a)*. Given that (i) *lead time* = 2 *months*, (ii) *prescribed time interval* (t_p) = *one month*, and (iii) *units are discrete*.

Let z be the number of trained hostesses available to meet the demand of 2 months lead time and the one month period. The girls admitted after one month (*i.e.* middle of lead time) are available for flying duties after completion of 2 months period (*i.e.* after 3 months from now). Thus, as seen from *Figs. 21.6, 21.7,* and *21.8,* there are three situations in all :

(i) $0 \le y \le z, 0 < x \le z - y$; *(ii)* $0 \le y \le z, x > z - y$; and *(iii)* $y \le z, x \ge 0$.

Hence proceeding as in *Model VIII (a),* we get the cost equation similar to eqn. (21.22).

21.6. MULTI-PERIOD PROBABILISTIC MODEL WITH CONSTANT LEAD TIME

In the deterministic models, it was assumed that demand occurred at constant rate and the lead time was either assumed zero or known exactly. But, here we shall discuss the case of probabilistic demand with constant lead time.

Obviously, a shortage occurs when the consumption of goods is more than that it was expected. So in order to avoid such situation of stockout the only alternative is to raise the reorder level; equivalently, maintaining thereby a safety stock. Thus, in order to determine an optimal level of safety stock, it is necessary to balance the cost of carrying this safety stock and the shortage (stockout) cost.

21.6–1. Model IX : Fixed-Order System

In this system, the inventory position is reviewed continuously and maintained upto a prescribed level. As and when the inventory level reaches the reorder level, an order is placed for fixed quantity q, which is nearly equal to the economic order quantity. This ordered quantity will be received at the end of the lead time period (L). As soon as the new supply is received, the cycle becomes complete. In the *Fig.* 21.10 below three such cycles are demonstrated.

First Cycle. As shown in the *Figure* 21.10, during the first cycle, the demand rate and lead time both have their average values resulting that the replenishment is received just when the inventory level touches the safety stock level.

Second Cycle. In the second cycle, the lead time (L) is slightly longer and the demand cuts into the safety stock before the replenishment is received.

Third Cycle. In this cycle, the lead time (L) is much longer which results not only in a cutting into the safety stock but also in a shortage (stockout) just before the receipt of replenishment.

Since the demand is probabilistic, it may cause the possibility of either *shortages* or *surpluses*. If the demand during the lead time L is more than the reorder level quantity (r), then a shortage of ($d_L - r$) quantity units will exist, otherwise, a surplus of ($r - d_L$) quantity units will remain in the inventory system.

In the case of shortages, there will be a shortage of a

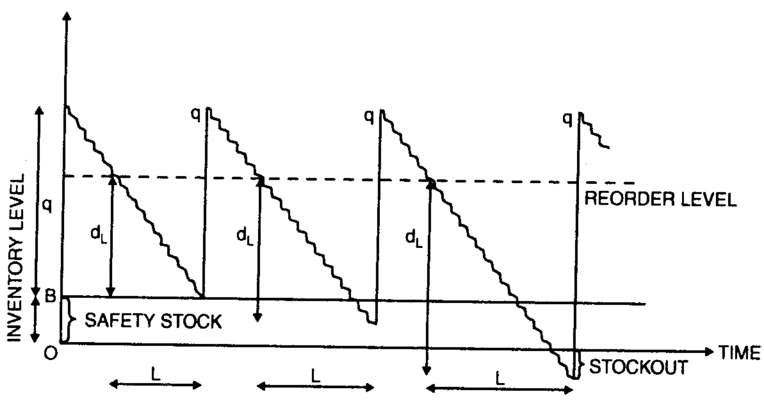


Fig. 21.10.
Probabilistic demand system with different constant lead time in each cycle.

cost (C_2) per item. On the other hand, in the case of a surplus, there will be an additional annual carrying cost (C_1) per item. Thus we have

Total cost = Ordering cost (C_3) + (actual + additional) carrying cost (C_1) + shortage cost (C_2)

As defined earlier,

$$\textit{Ordering cost} = \frac{D}{q} C_3 \quad ...(21.31) \qquad \textit{Carrying cost} = \frac{q}{2} C_1 \quad ...(21.32)$$

Since the size of the shortage quantity is $(d_L - r)$ items, total shortage cost = $(d_L - r) C_2$.

Also, since the demand during lead time is probabilistic, the expected shortage cost per cycle becomes :

$$= C_2 \Sigma (d_L - r) p(d),$$

where $p(d)$ is the probability for several possible levels of demand (d_L) with D/q number of cycles per year, the expected annual cost will be

$$= C_2 (D/q) \Sigma (d_L - r) p(d). \quad ...(21.33)$$

In order to avoid the chances of shortage, the safety stock (B) is introduced in the system. The safety stock (B) is the difference between reorder level quantity (r) and the expected lead time demand (d_L), *i.e.*

$$\textit{Safety stock } (B) = (r - d_L).$$

Thus safety stock is made-up by increasing the quantity (r) of the reorder point for the expected lead time (d_L). The increased carrying cost for maintaining a safety stock against shortages is given by $C_1(r - d_L)$. Now combining this cost with the canying cost $(q/2) C_1$, we get

$$\textit{Total carrying cost} = \tfrac{1}{2}q\, C_1 + C_1 (r - d_L) = C_1 [(q/2) + (r - d_L)] \quad ...(21.34)$$

Now combining the costs given by (21.31), (21.33) and (21.34), the total expected inventory cost becomes

$$C(q, r) = C_3 (D/q) + C_1 (q/2 + r - d_L) + C_2 (D/q) \Sigma (d_L - r) p(d). \quad ...(21.35)$$

Our decision policy is to find the values of the decision variables (q and r) that will make the new total inventory cost as minimum as possible. The decision variables (q and r) are interdependent and accurate value of these variables must be derived simultaneously from the total expected cost equation (21.35). But, in order to avoid complexity we assume that q and r are independent of each other.

To obtain the good approximation of the optimal order quanitity (q^*) we use the EOQ formula that considers the cycle costs only. The value of q^* thus obtained can be used to obtain the value of r to minimize safety stock costs.

The level of safety stock can be defined by determining the reorder level (r). The alternative values of r can be obtained in terms of the total cost formula including the cost of carrying safety stock + the expected costs of shortages. This formula can be put in the form :

$$C = C_1 (r - d_L) + C_2 (D/q^*) \Sigma (d_L - r) p(d) \quad ...(21.36)$$

or $$\text{Total cost} = \begin{pmatrix}\text{Carrying costs}\\ \text{of safety stock}\\ \text{per period}\end{pmatrix} . \begin{pmatrix}\text{Safety}\\ \text{stock}\\ \text{level}\end{pmatrix} + \begin{pmatrix}\text{Cost}\\ \text{of a}\\ \text{shortage}\end{pmatrix} . \begin{pmatrix}\text{Number of}\\ \text{reorder}\\ \text{cycles}\end{pmatrix} . \begin{pmatrix}\text{Expected units}\\ \text{short per}\\ \text{reorder cycle}\end{pmatrix}$$

Q. What are the disadvantages of fixed-orders-quantity system ? Explain how these disadvantages are overcome by fixed-interval system. **[Delhi (MBA) 95]**

Illustrative Example

Example 11. *A certain item has an annual demand of 2000 units. The cost of placing an order is Rs. 400 and the annual carrying cost is Rs. 10 per unit. The costs of stockout are estimated to average Rs. 10. The demand during lead time tends to be randomly distributed throughout the year, so that a Poisson distribution may be assumed. There are 250 working days per year and lead time is 5 working days.*

Demand during lead time :	70	75	80	85	90	95	100
Probability :	0.02	0.14	0.23	0.24	0.21	0.12	0.04

Determine the optimal order quantity and reorder level.

Solution. We are given that

$D =$ 2000 units/year, $C_3 =$ Rs. 400/order

$C_1 =$ Rs. 10 per unit/per year, $C_2 =$ Rs. 10 per unit, and *lead time* $(L) = 5$ days.

The optimal value of order quantity can be obtained as:

$$q^* = \sqrt{\left(\frac{2DC_3}{C_1}\right)} = \sqrt{\frac{2 \times 2000 \times 400}{10}} = 400 \text{ units.}$$

The expected number of units *demanded* (denoted by d_L) during the lead time is given by

$$D_L = d_{L_1} p(d_{L_1}) + d_{L_2} p(d_{L_2}) + d_{L_3} p(d_{L_3}) + d_{L_4} p(d_{L_4}) + d_{L_5} p(d_{L_5}) + d_{L_6} p(d_{L_6}) + d_{L_7} p(d_{L_7})$$
$$= (70 \times .02) + (75 \times .14) + (80 \times .23) + (85 \times .24) + (90 \times .21) + (95 \times .12) \times (100 \times .04)$$
$$= 85 \text{ expected units.}$$

If the safety stock is not provided, then shortages will occur whenever the demand during lead time exceed 85 units. The expected number of shortage units per reorder cycle are computed in the following table.

Reorder level (r)	Safety stock level (B)	Lead time demand (d_L)	Shortages during lead time $(d_L - r)$	Prob. of demand during L $[p(d_L)]$	Expected shortage during L $(d_L - r)$ $p(d_L)$	Expected units short per reorder cycle $\Sigma(d_L - r)\, p(d_L)$
85	0	85	0	0.24	0	2.85
		90	5	0.21	1.05	
		95	10	0.12	1.20	
		100	15	0.04	0.60	
90	5	90	0	0.21	0	1.0
		95	5	0.12	0.6	
		100	10	0.04	0.4	
95	10	95	0	0.12	0	0.2
		100	5	0.04	0.2	
100	15	100	0	0.04	0	0

From above table it may be observed that whenever there is no safety stock, the expected number of units short per reorder cycle are 2.85. But, when the buffer stock of 5, 10, and 15 units is provided, the expected shortage is 1.0, 0.2 and 0, respectively.

The annual carrying costs due to safety stock is 0, 50, 100 and 150 at the rate of Rs. 10 per unit for safety stock ranging from 0 to 15 units. These costs and the expected annual shortage cost is computed as given in the following table.

Safety stock level (B)	Expected shortage per reorder cycle $\Sigma\,(d_L - r)\, p(d_L)$	Per unit shortage cost (C_2)	Reorder cycle per year (D/q^*)	Expected annual shortage cost $C_2\,(D/q^*)\,\Sigma\,(d_L - r)\, p(d_L)$
0	2.85	10	5	142.5
5	1.00	10	5	50.0
10	0.20	10	5	10.0
15	0	10	5	0

The carrying costs of various safety stock levels and expected annual shortage cost are combined together in the following table to show the inventory trade-off between the two.

Safety stock level	Safety stock carrying cost	Expected annual shortage cost	Total annual expected cost
0	0	142.5	142.5
5	50	50.0	100.0
10	100	10.0	110.0
15	150	0	150.0

From this table we observe that total annual expected cost reaches a minimum level at the safety stock level of 5 units. Hence, optimal reorder level, $r = B + d_L = 5 + 85 = 90$ units.

21.7. ADVANTAGES OF INVENTORY CONTROL

The main advantages of inventory control are :

1. Inventory control ensures an adequate supply of items to customers and avoids the shortages as far as possible at the minimum cost.
2. It makes use of available capital (and/or storage space) in a most effective way and avoids an unnecessary expenditure on high inventory etc.
3. The risk of loss due to change in prices of items is reduced.
4. It ensures a smooth and efficient running of the organization.
5. It provides advantages of quality discounts on bulk purchases.
6. It serves as a buffer stock required due to delay in supply to the market.
7. It eliminates the possibility of duplicate ordering.
8. It helps to minimize the loss due to deterioration, obsolescence, damages or pilferage, etc.
9. It helps in maintaining the economy by absorbing some of the fluctuations when the demand for an item fluctuates or is seasonal.
10. It minimizes and controls accumulation and build-up of surplus stock, and removes the dead movable surplus stock as far as possible.
11. It utilizes the benefits of price fluctuations. Thus, it may be concluded that with the help of a good inventory, a firm is able to make purchases in economic lots, to maintain continuity of operations, to avoid small time consuming orders and to guarantee prompt delivery of finished goods.

EXAMINATION PROBLEMS

1. If the demand for a certain product has a rectangular distribution between 5000 and 6000, find the optimal order quantity if storage cost is Rs. 10 per unit upto 5200 units and Rs. 9.00 per unit above 5200 units and shortage cost is Rs. 20.00 per unit.
2. The probability distribution of the demand for a product has been estimated to be :

Sales :	0	1	2	3	4	5	6
Probability :	0.05	0.15	0.30	0.35	0.10	0.05	0.00

Each unit sells for Rs. 100, and the total cost per unit is Rs. 60. If the product is not sold, it is completely worthless.

(i) Assuming no re-ordering is possible, how many units should be purchased ?

(ii) If the customer will is estimated to be Rs. 65 for every unit for which there is unfilled demand, how many units should be ordered ?

3. The probability distribution of the demand for a certain item is as follows :

Sales :	0	1	2	3	4	5
Prob. :	0.05	0.35	0.08	0.15	0.20	0.17

The cost per unit is Rs. 60 and it sells for Rs. 100. If the product is not sold, it is completely worthless. What optimum policy would you recommend.
[**Hint.** Use *Model VI(a)*]

4. The probability distribution of monthly sales of a certain item is as follows :

Monthly sales :	0	1	2	3	4	5	6
Probability :	0.01	0.06	0.25	0.35	0.20	0.03	0.10

The cost of carrying inventory is Rs. 30 per unit per month and the cost of unit short is Rs. 70 per month. Determine the optimum stock level which minimizes the total expected cost.
[**Hint.** C_1 = Rs. 30, C_2 = Rs. 70. Use the result (3.19) of *Model VII (b)*.]
[**Ans.** Monthly sale of given items is 3.]

5. Cost of a spare item is Rs. 10,000 per unit. The estimated cost of non-availability of spares when needed (because of a failure of the part in use) is Rs. 2,00,000. The estimated requirement of this spare during life time of 20 years of the installation is summarized below :

Number of spares needed in 20 years :	0	1	2	3	4	5	6 or more
Estimated Probability :	0.90	0.05	0.02	0.01	0.01	0.01	0.00

What is the optimum stock level ?

6. Accounting records and market forecasts provide the basis for the following information regarding a product under inventory management :
Annual demand = 1200, annual carrying cost = Rs. 16.00, ordering cost = Rs. 24.00, stock out cost = Rs. 40.00, lead time = 10 days, working days per year = 300.

Demand during lead time :	38	39	40	41	42
Probability :	0.10	0.20	0.38	0.24	0.08

Determine the decision variables for proper inventory management.

7. A manufacturer wants to know what is the optimum stock level of a certain part, which is used in filling orders which occurs at a relatively constant rate, but not of constant size. The delivery of these parts is made to him immediately. He regularly places x orders for the parts at the beginning of each month. The prob. of demands of these parts are given below. The cost of holding a unit in stock per month is Re. 1.00 and shortage cost is Rs. 20 per month :

Number of parts required (r) :	0	1	2	3	4	5	6
Prob. of requirements $p(r)$:	1	.2	.2	.3	.1	.1	.0

[Meerut M.Sc. (Math.) BP-96]

II -Selective Inventory Management

Inventories in an organization can be controlled in various ways. For example, one method is to keep an up-to-date record of receipt and issue of inventory items. This method is known as ***perpetual inventory system.*** Another method is to find the optimum order quantity of each item to be stocked. Thus economic order quantity (EOQ) is based on the rate of demand, lead time for replenishment, and related costs. The EOQ models have already been discussed in the previous chapter.

But, we know that every organization consumes several items of store. Since all the items are not of equal importance, a high degree control on inventories of each item is neither applicable nor useful. So it becomes necessary to classify the items in groups depending upon their utility importance. Such type of classification is named as the *principle of selective control* which is applied to control the inventories. Here we shall discuss the techniques of selective control in the following sections.

21.8. ALWAYS BETTER CONTROL (A B C) ANALYSIS [IGNOU 2001]

'*ABC*' Analysis is a basic tool which helps the management to place their efforts where the results would be useful to the greatest possible extent. The first important step in inventory management is to have a selective approach to fix-up inventory levels, order quantities, and the extent to which the control can be exercised. The selective approach mainly depends on the annual consumption of various items.

For example, the items like nuts and bolts (though being equally important) cost less than the items like engines. But we cannot safely stock the items like engines because of their heavy cost, while the items like nut-bolts can be easily stocked. Thus, less control is required for stocking the items like nut-bolts etc. But, more emphasis should be given to control the stocking of big items like engines. The investment of such items is substantial, and record keeping is expensive.

ABC (*Always Better Control*) analysis is a very effective tool for such selective control. This technique involves the classification of inventory items into three categories *A*, *B* and *C* in descending order of annual consumption and annual monetary value of each item. Based on *ABC* analysis, an average pattern of percentages of items and percentages of their annual consumption value may be planned as below :

Category	*Percentage of Items (%)*	*Percentage of Annual Usage (%)*
A	10	80
B	20	15
C	0	5

In practice it is experienced that a bulk of items in an inventory have low usage value.

$$Annual\ usage\ value = (annual\ requirement) \times per\ unit\ cost$$

Thus for better and more economic control of items in inventory, the items should be classified according to their significance or priority for recording. So for effective inventory control a decision has to be made that—which items are little things and which need more careful control. The items of an inventory can be classified according to the following characteristics.

(*i*) Items which are functionally critical to the operations, no matter how little they cost.

(*ii*) Items that are important because their usage value is very high.

(*iii*) Items having average usage value.

(*iv*) Items which have low usage value.

The '*ABC*' analysis is based on **Pareto's Law** that—*a few high usage value items constitute a major part of the capital invested in inventories, where as bulk of items in inventory having low usage value constitute insignificant part of the capital.*

This concept is based upon selective control. If there are large number of items to be analysed, then sampling technique may be used for *ABC* analysis.

In *ABC* analysis, the items are classified in three main categories based on their respective usage value :

(i) Category '*A*' items. More costly and valuable items are classified as '*A*'. Such items have large investment but not much in number, *e.g.*, say 10% of items account for 75% of total capital invested in inventory. So, more careful and closer control is needed for such items.

The items of this category should be ordered frequently but in small number. A periodic review policy should be followed to minimize the shortage percentage of such items and top inventory staff should control these items. These items have high carrying cost and frequent orders of smaller size for these items can result in enormous savings.

(ii) Category '*B*' items. The items having average consumption value are classified as '*B*'. Nearly 15% of the items in an inventory account for 15% for the total investment. These items have less importance than '*A*' class items, but are much costly to pay more attention on their use. These items cannot be overlooked and require lesser degree of control than those in category '*A*'. Statistical sampling is generally useful to control them.

(iii) Category '*C*' items. The items having low consumption value are put in category '*C*'. Nearly 75% of inventory items account only for 10% of the total invested capital. Such items can be stocked at an operative place where people can help themselves with any requisition formality. These items can be charged to an over head account. In fact, loose control of '*C*' items increase their investment cost and expenditure on shelf-wear, obsolescence and wasteful use, but this will not be so much offset for the saving in recording costs.

Important points for *ABC* analysis :

(i) *Whenever the items can be substituted for each other they should be preferably considered as one item.*

(ii) *More emphasis should be given to the value of consumption and not to the cost per unit of item.*

(iii) *While classifying as A, B or C, all the items consumed by an organization should be considerd together, instead of considering like : spares, raw materials, semi-finished, and finished items, and then classifying as A, B or C.*

(iv) If required, there may be *more than three* clases and period of consumption need necessarily be one year.

The comparison of items in *A*, *B* and *C* categories can be presented in the following tabular form :

	Class 'A' Items	*Class 'B' Items*	*Class 'C' Items*
1.	Close control is required.	Modrate control is required.	Loose control is required.
2.	Size of order is based on calculated requirement.	Size of order is based on their consumption.	Size of order is based on the level of inventory.
3.	Procured from many sources.	Procured from two or three sources.	Procured from two sources.
4.	Requires keeping records of receipt and consumption.	Also, requires keeping records of receipt and consumption.	No need of keeping any records.
5.	More effort is made to reduce lead time.	Moderate effort is made to reduce lead time.	Minimum effort is made to reduce lead time.
6.	Close checks on schedule revision is required.	Some checks on changes are required on need.	No checks are required against any need.
7.	Frequent ordering is required.	Less frequent ordering is required.	Bulk ordering is required.
8.	Continual expediting.	Expediting for prospective shortages.	No expediting.
9.	Accurate forecasts.	Less accurate forecasts.	Approximate forecasts.
10.	Low safety stock for less than two weeks.	Large safety stock upto two to three months.	Large safety stock for more than three months.
11.	Have high consumption value.	Have average consumption value.	Have low consumption value.

With '*ABC*' control technique, it is also possible to reduce the investment in inventories as seen in the following example.

Example 12. A company making no use of '*ABC*' control technique for its inventory makes 4 orders a year in respect of each item to get three months' supply of every item. Considering a sample of 3 items with different level of annual consumption, their average inventory (which is equal to one half of the order quantity) is computed in the following table.

Item	*Annual consumption (Rs.)*	*Number of orders*	*Average working inventory*
A	40,000	4	10,000/2 or 5,000
B	4,000	4	1,000/2 or 500
C	400	4	100/2 or 50
Total	Rs. 44,400	12	5550

But, by keeping the same number of orders per year, *i.e.* 12, inventories can be reduced by 39% merely by segregating (separating) items according to their consumption value (*ABC* analysis) which is demonstrated in the following table.

Item	*Annual consumption (Rs.)*	*Number of orders*	*Average working inventory*
A	40,000	8	5,000/2 or 2,500
B	4,000	3	1,333/2 or 667
C	400	1	400/2 or 200
Total	Rs. 44,400	12	3367

Thus we observe that the inventory investment is reduced by 39%.

Step-by-Step Procedure for '*ABC*' Analysis :

Following are the *steps* for classification of items into '*A*', '*B*' or '*C*' category.

Step 1. Determine the number of units sold or used in the past one year (= 12 months) period.

Step 2. Determine the unit cost standard for each item.

Step 3. Compute the annual usage value (in Rs.) of each item consumed by multiplying the annual consumption (of units) by its unit price.

Step 4. Arrange the items in a decending order according to their respective usage value computed in *step* 3.

Step 5. Prepare a table showing unit cost, annual consumption and annual usage value for each item.

Step 6. Calculate the cumulative sum of the number of items and the usage value for each item obtained in *step* 3.

Step 7. Find the percentage of the values obtained in *step* 6 with respect to the grand total of the corresponding columns.

Step 8. Draw a graph by taking % of items on X-axis and the corresponding usage value % on Y-axis. After plotting various points on the graph we draw a smooth curve as shown in *Fig.* 21.11 (a).

Step 9. Mark the points *X* and *Y* where the slope of the curve changes sharply (such points are called the *points of inflexion*).

Step 10. Finally, the usage value and the % of items corresponding to these points determine the classification of items as *A*, *B* or *C*.

This classification can also be represented graphically as shown in Fig. 21.11 (*b*).

21.8–1. Limitations of *ABC* Analysis

(*i*) *ABC* analysis does not permit precise consideration of all relevent problems of inventory control. For example, a never-ending problem in inventory management is that of adequately handling thousands of low-value '*C*' items. Low value purchases frequently require more items, and consequently reduce the time allowance available and purchasing personnel for value analysis, vendor investigation, and other '*B*' items.

(*ii*) If *ABC* analysis is not updated and reviewed periodically, the real purpose of control may be defeated. For example, '*C*' items like diesel oil in a firm, will become most high-value items during power crisis, and therefore should require more attention.

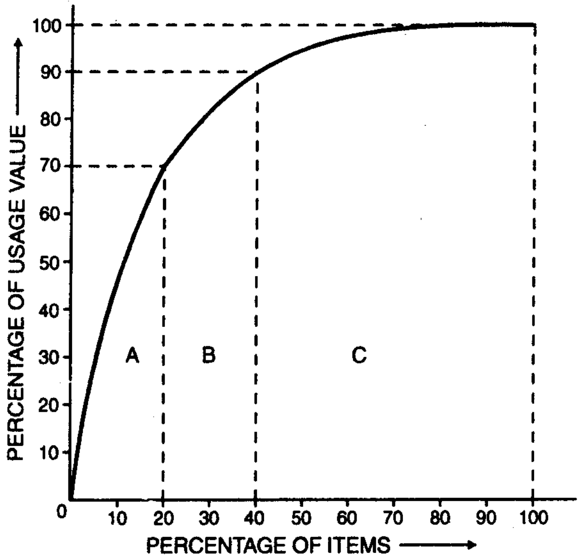


Fig. 21.11 (a)

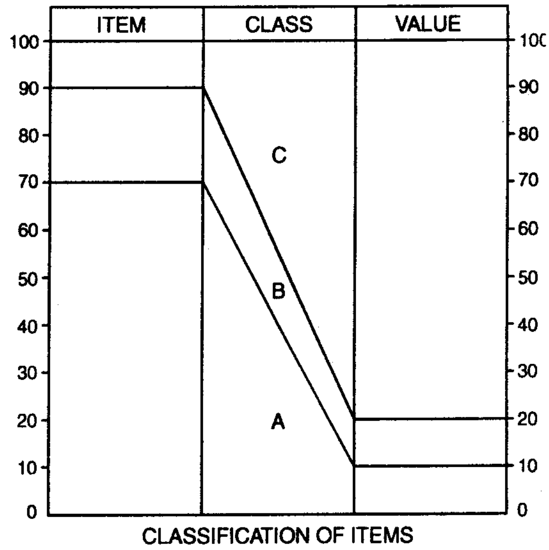


Fig. 21.11 (b)

(iii) The periodic consumption value (not the unit value) is the basis for *ABC* classification. Hence *ABC* classification can lead to overlooking the needs of spare parts whose criticality is high but consumption value is low.

Q. 1. Explain '*ABC*' analysis. What are its advantages and limitations, if any ? **[JNTU (B. Tech.) 98]**

2. What is the '*ABC*' analysis ? Why is it necessary ? What are the basic steps in implementing it ?
3. Explain the importance of '*ABC*' analysis in the problem of inventory control of an organization using a large number of items.
4. Describe the norms you would use for controlling inventories classified by *ABC* analysis.
5. Explain briefly *ABC* analysis as applied to procurement policy.
   "Purchasing Management should shoulder special responsibility for '*A*' items and the '*A*' items should not be handled on any routine procurement policy"—Discuss.
6. Explain all the levels of analysis for effective control over inventories. Provide details for ABC analysis with special reference to method of classification. **[Delhi M.Sc. (OR). 92, 90]**

21.8–2. Illustrative Examples

Example 13. *The following information is known about a group of items. Classify the material in A, B, C, classification.*

Model Number	*Annual Consumption in pieces*	*Unit Price (in Paice)*
501	30,000	10
502	2,80,000	15
503	3,000	10
504	1,10,000	5
505	4,000	5
506	2,20,000	10
507	15,000	5
508	80,000	5
509	60,000	15
510	8,000	10

Solution. The number of items sold in the past 12 months as well as the unit cost standard for each item are given in the problem. Now multiplying annual consumption of each item by its unit cost and then ranking the items in the descending order of the usage values thus obtained, the following table is obtained :

Model Number	*Annual Consumption (in pieces)*	*Unit Price (in Paise)*	*Usage value (in Rs.)*	*Ranking*
(1)	(2)	(3)	(4) = (3) × (2)	(5)
501	30,000	10	3,000	6
502	2,80,000	15	42,000	1
503	3,000	10	300	9
504	1,10,000	5	5,500	4
505	4,000	5	200	10
506	2,20,000	10	22,000	2
507	15,000	5	750	8
508	80,000	5	4,000	5
509	60,000	15	9,000	3
510	8,000	10	800	7

Now compute the cumulative total number of items and their usage values and convert the accumulated total items of percentage of the grand total. The following *ABC* classification is thus obtained.

Rank	*Model No.*	*% of items*	*Cumulative Usage Value (Rs.)*	*Cumulative % in (Rs.)*	*Category*
(1)	(2)	(3)	(4)	(5)	(6)
1	502	10	42,000	48.0	A
2	506	20	64,000	73.0	
3	509	30	73,000	83.0	
4	504	40	78,500	90.0	B
5	508	50	82,500	94.0	
6	501	60	85,500	98.0	
7	510	70	86,300	98.6	
8	507	80	87,050	94.4	C
9	503	90	86,350	99.6	
10	505	100	87,550	100	

The cut-off points are determined from the following graph.

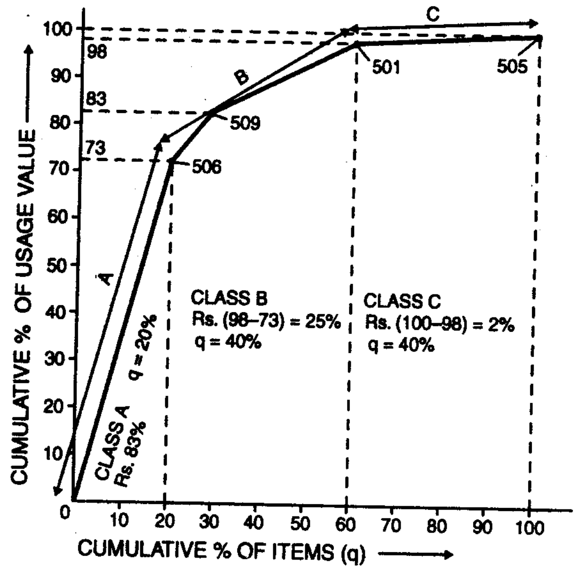


Fig. 21.12

Example 14. *Perform ABC analysis on the following sample of items in an ivnentory :*

Item name	*Annual consumpton*	*Price per unit (in Paise)*
A	300	10
B	2,800	15
C	30	10
D	1,100	5
E	40	5
F	220	100
G	1,500	5
H	800	5
I	600	15
J	80	10

[Meerut (M. Com.) Jan. 98 BP]

Solution. ***Setp* 1.** The usage value of the items can be calculated in the following tabular form by multiplying the annual consumption and their corresponding unit price.

Items :	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Usage value (Rs.) :	30	420	3	55	2	220	75	40	90	8

***Step* 2.** The usage values are ranked in descending order and the cumulative percentages of the number of items and usage values are determined.

Item (1)	*Usage value in descending order* (2)	*Cumulative number of items* (3)	*% of number of items* (4)	*Cumulative usage value* (5)	*% of cumulative usage value* (6)
B	420	1	10	420	44.50
F	220	2	20	640	67.79
I	90	3	30	730	77.41
G	75	4	40	805	85.37
D	55	5	50	860	91.20
H	40	6	60	900	95.44
A	30	7	70	930	98.62
J	8	8	80	938	99.47
C	3	9	90	941	99.79
E	2	10	100	943	100.00

***Step* 3.** The % of items and the % of usage value can be plotted on the graph paper as follows :

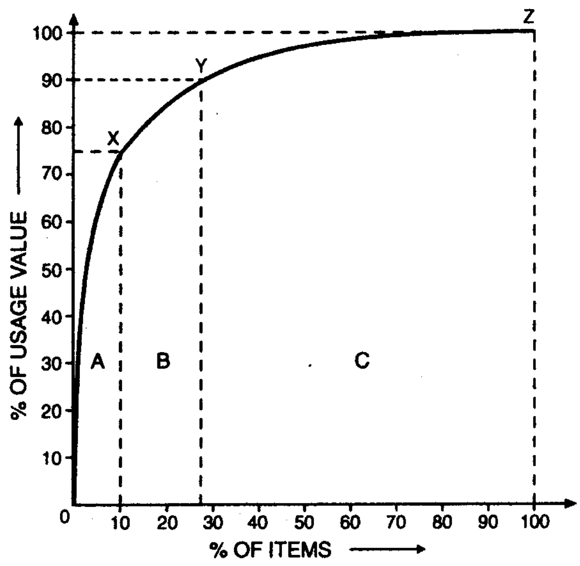


Fig. 21.13.

***Step* 4. (Observation).** It can be observed that the items lying in classes *A, B* and *C* are :

Class of Items	*name of Items*	*% of usage value*	*% of Items*
A	B and F	67.49	20
B	I and G	17.58	20
C	D, H, A, J, C, E	14.63	60

21.9. VITAL, ESSENTIAL, AND DESIRABLE (V E D) ANALYSIS

The *VED* analysis is based on the *criticality* of the items. If the items are arranged in the descending order of their *criticality viz.*

V → *Vital items, E* → *Essential items, D* → *Desirable items,*

then the more attention is paid to the *V*-type items.

The criticality of an item may be of two types : (i) *Technical,* (ii) *Environmental.*

Vital Items. Such items are those which when required, and not available, they make the whole system in-operative.

For example, a clutch-wire is *vital item* for the speeding vehicle, like scooter, motor-cycle etc.

Essential Items. Those items which when demanded, and not available reduce the efficiency of the system are called *essential items.* For example, telephone is an essential item. In the case of *E-items* some risk can be taken.

Desirable Items. Desirable items are such that—even if they are not available, they neither stop the system nor reduce its efficiency, but it will be good if they are present in the system.

The *VED* analysis is useful in stock-controlling of spare parts required for maintenance. This analysis can also be very useful to capital intensive process industries. Since this analysis is based on the criticality of an item, it can be used in the case of such raw materials which are rarely available.

Moreover, the *ABC* and *VED* both can be combined to control the stocking of spare parts. The various control actions of *ABC–VED* combined can be summarized in the following table :

ABC Classification	*VED Classification*		
	V	*E*	*D*
A	Regular stock with constant control	Medium stock	No stock
B	Medium stock	Medium stock	Very low stock
C	High stock	Medium stock	Low stock

Q. Write a short note on VED analysis.

21.10. *XYZ*-ANALYSIS BASED ON INVENTORY VALUE

As we have seen that the basis for *ABC* classification was the consumption value of the items. But, the basis of the *XYZ*-classification is the closing *Inventory value* of the items, *viz.,*

X → Items with high inventory value

Y → Items with moderate inventory value

Z → Items with low inventory value

XYZ–classification is usually performed once in a year during the annual stock-taking device. This analysis helps us in identifying the items which are being stocked extensively.

XYZ-classification can also be combined with the *ABC*-analysis and the controls on the items can be affected as shown in the table below :

ABC Classification	*XYZ-Classification*		
	X	*Y*	*Z*
A	Attempt to reduce the stock	Attempt to convert Z-Items	Items are within control
B	Review stock and consumption more often	Items are with in control	Review bi-annually
C	Dispose off the surplus items	Check and maintain the control	Review annually.

21.11. *FNSD*-ANALYSIS BASED ON USAGE RATE OF ITEMS

The items can also be classified into categories according to descending order, of their *usage rate,* or *movement* as follows :

F → Fast moving items, *N* → Normal moving items, *S* → Slow moving items, *D* → Dead items.

In this classification, a close attention is paid to the *F*-items, while *D*-items are transferred to the disposal cell.

FNSD-analysis is particularly useful to combat obsolete items. The cut-off points of four classes are usually indicated in terms of the number of issues in the past two to three years. For example, no issue in the past two years, may be classified in *D*-category; upto 10 issues during that period in *S*-category; upto 20 issues in *N*-category and more than 20 issues are put in *F*-category.

The *XYZ* and *FNSD* both can be combined to control the obsolete items, which is useful in the timely prevention of obsolescence.

Q. 1. Explain the basis of selective inventory control and state the different selection techniques adopted in Inventory control system. Give a brief note on each.

21.12. SUMMARY OF SELECTIVE CONTROL TECHNIQUES

The suitablility of selective control techniques (*ABC, VED, XYZ, FNSD*) depends upon the nature of inventories carried by an organization. For example, a spares consuming organization, like Indian Oil, would rely more on *VED* classification. As discussed earlier, sometimes it is more useful to combine two selective control techniques. Because, the *XYZ-FNSD* combine reduces obsolescence while *ABC-VED* combine is useful in the case of spare parts, etc.

We may summarize the four selective control techniques with their basis of classification and main uses :

Selective Control Technique	*Basis of Classification*	*Main Uses*
ABC	Consumption value	Controlling raw material components and work-in-progress inventories
VED	Criticality of items	Determining the inventory levels of spare parts.
XYZ	Value of items in storage	Reviewing the inventories and other uses.
FNSD	Consumption rate (or movement) of items	Controlling obsolescence.

Q. 1. **(a)** Define "sevice level" used in inventory control.

(b) Is the service level different from probability of not facing shortage ? Why ?

(c) Does the service level depend on the class of the item, when *ABC* and *VED* classification of items are used ? Why ?

(d) If the demand during lead time is varying and no buffer stock is to be maintained under what condition would the service level become non-zero ?

EXAMINATION PROBLEMS

1. A company purchases three items *A, B* and *C*. Their annual demand and unit prices are given in the following table.

Items	*Annual Demand (units)*	*Unit Price (Rs.)*
A	1,00,000	3
B	80,000	2
C	600	96

If the company wants to place forty orders per year for all the three items, what is the optimal number of orders for each item ?

2. The following thirty numbers represent the annual value in thousand of rupees of some thirty items of materials selected at random. Carry-out an *ABC* analysis and list out the values of '*A*' items only.

1	2	4	9	75	4
3	6	13	2	4	12
100	2	7	40	15	55
1	11	25	15	8	10
1	20	30	1	3	5

3. Make an *ABC* analysis for the following items in a store and construct the *ABC* Analysis Chart :

Item No.	:	1	2	3	4	5	6	7	8	9	10
Annual consumption (Rs.)	:	1,100	2,000	15,000	1,485	3,222	4,987	10,000	8,200	7,525	6,221
Item No.	:	11	12	13	14	15	16	17	18	19	20
Annual consumption (Rs.)	:	1,242	4,320	5,200	400	525	2,382	3,725	1,020	2,100	1,710
Item No.	:	21	22	23	24	25	26	27	28	29	30
Annual consumption (Rs.)	:	578	982	825	102	9,270	11,000	98	92	1,010	1,200
Item No.	:	31	32	33	34	35	36	37			
Annual consumption (Rs.)	:	1,450	8,900	902	12,000	100	425	9,321			

4. What is *ABC* Analysis ? For what purpose do the Inventory Managers use *ABC* Analysis ? Explain the usage of *ABC* analysis to various functional areas. Classify the following 14 items in *ABC* categories :

Code No.	*Monthly Consumption (in Rs.)*
D–179–0	451
D–115–0	1,052
D–186–0	205
D–191	893
D–192	843
D–193	727
D–195	412

Code No.	*Monthly Consumption (in Rs.)*
D–196	214
D–198–0	188
D–199	172
D–200	170
D–204	5,056
D–205	159
D–212	3,424

How the policies with regard to safety stocks, order quantity, materials control and inventory system will be different for the items classified as *A*, *B* and *C* ?

5. What is selective inventory control ? From the following details, draw a plan of *ABC* selective control.

Item	:	1	2	3	4	5	6	7	8	9	10	11	12
Unit C ('000)	:	7	24	1.5	0.6	38	40	60	3	0.3	29	11.5	4.1
Unit cost	:	5	3	10	22	1.5	0.5	0.2	3.5	8	8.4	7.1	6.2

21.13. EXPONENTIAL SMOOTHING METHOD

The exponential smoothing method determines a forecast of sales for the next period by obtaining a weighted average of sales in the current period and the sales forecast made in the current period. The sum of all weighted values must be equal to one. For example, if the weight for actual current sale is taken at 0.1, then the weight for the forecast made in the current period would have to be 0.9. So the selection of the weighted factor (A) is of great importance. By experience it has been found that setting *A* at 2 gives more favourable results. This setting smoothes the extremes of current sales while allowing for definite fluctuations in the trend of sales.

The basic exponential smoothing formula is given by :

$$\begin{pmatrix}\text{new sales forecast}\\ \text{for the coming}\\ \text{period}\end{pmatrix} = A\begin{pmatrix}\text{actual sales}\\ \text{during the}\\ \text{current period}\end{pmatrix} + (1-A)\begin{pmatrix}\text{forecasted sales}\\ \text{determined in the}\\ \text{current period}\end{pmatrix}$$

This formula is written, mathematically, as follows :

$$S_t' = AS_t + (1 - A)\, S_{t-1} \qquad \text{...(21.37)}$$

where $0 \le A \le 1$ and,

$S_t' \rightarrow$ new sales forecast (in units) for the coming period, made at the end of period t.

$A \rightarrow$ the weighted factor, it is some number between 0 and 1.

$S_t \rightarrow$ actual sales for period t.

$S'_{t-1} \rightarrow$ sales forecast for period 't' made in period $t-1$, *i.e.*, one period before it.

For equation (21.37) it is assumed that average sales over the year will remain approximately constant or there will be no upward or downward trend and no seasonal influence. But, these assumptions are unrealistic. So, the forecast method must be adjusted for both.

The seasonal influences may be independent or may bear direct relationship to volumes of sales. The assumption of the relationship of seasonal amplitude being dependent upon the sales volume is moe common, so the updating of above formula is very essential.

Seasonal factors, where B is the seasonal smoothing parameter, are up-dated according to the formula :

$$F_t = B\frac{S_t}{S_t'} + (1-B)\,F_{t-N} \qquad \text{...(21.38)}$$

where $0 \le B \le 1$ and,

$F_t \rightarrow$ seasonal factor for period t (ratio of actual sales to smoothed sales).

$B \rightarrow$ seasonal smoothing factor, which is some number between 0 and 1.

$S_t \rightarrow$ actual sales for period t.

$S_t' \rightarrow$ non-seasonal sales forecast for the coming period, made at the end of period t.

$N \rightarrow$ number of periods for seasonal effect (ordinarily 12 months).

If there is rising trend, then the trend adjustment adds a small amount to each forecast. While a small amount is subtracted if there is a declining trend in sales. Experience with the trend weighting factor (C) for the firm has proved that better results are obtained when it approximates 'A'.

The basic trend adjustment formula is given by

$$R_t = C\,(S_t' - S'_{t-1}) + (1-C)\,R_{t-1}$$

where $0 \le C \le 1$ and,

$R_t \rightarrow$ the trend adjustment (in units)

$C \rightarrow$ trend weighting factor which is some number between 0 and 1.

$S'_{t-1} \rightarrow$ non-trend sales forecast for period t made in period $t-1$, *i.e.*, one period before t.

$S'_t \rightarrow$ non-trend sales forecast for the next-period, made at the end of period t.

$R_{t-1} \rightarrow$ trend adjustment for period t made in period $t-1$, *i.e.*, one period before it.

Now after solving above equations for the values of S_t', F_t, and R_t, the final step is to compute the revised forecasts (S_t'' is adjusted for seasonal and trend adjustments) by using the formula

$$S''_{t,T} = (S_t' + TR_t)\,F_{t-N+T} \qquad \text{...(21.39)}$$

where $T = 1, 2, \dots, N$.

Q. 1. A firm uses simple exponential smoothing with smoothing constant $\alpha = 0.1$ to forecast demand. The forecast for the week of February 1 was 500 units whereas, the actual demand turned out to be 450 units.

(i) Forecast the demand during the week of February 8.

(ii) Assume that the actual demand during the week of February 8 turned out to be 505 units. Forecast the demand for the week of February 15. Continue on forecasting through March 15 assuming that subsequent demands were actually 516, 488, 467, 554 and 500 units.

EXAMINATIONS REVIEW PROBLEMS

1. What are the objectives of inventory control ? Describe the method of carrying out *ABC* analysis and suggest a percentage of items, their consumption value and financial limits prescribed for *ABC* category in a typical organization.
2. State whether the following statements are correct. Give reasons :
   (i) Safety stock increases as demand increases.
   (ii) In *ABC* analysis high cost items are most likely to fall in the category *A*, and least cost items are likely to fall in category *C*.
   (iii) To protect against stockouts, a large batch size is a must.

(iv) EOQ is based on a balancing between inventory carrying costs and shortage costs.

(v) Lead time is the time interval elapsing between the placement of a replenishment order and the receipt of last instalment of goods against the order.

[Ans.

(i) *Not true,* becasue the safety stock varies with the fluctuation in demand and not with the level of demand.

(ii) *Not true,* because the items are classified on the basis of their annual usage value and not only on their cost.

(iii) *True,* because as the batch size rises (*i.e.,* more units per lot), the company orders fewer times in a year thus tne stock moves to lower points fewer times a year and the number of times the company is vulnerable to stock out diminishes.

(iv) *Not true,* because EOQ is the size of inventory order, which minimizes total annual cost of ordering and carrying inventory. It does not take into account the shortage costs at all.

(v) *Not true,* because the receipt of first instalment of goods marks the terminal point of the lead time and not the receipt of last instalment of goods against the order.]

3. Discuss the problem of inventory control when stochastic demand is uniform, production of commodity is instantaneous and lead time is negligible.

4. Solve the inventory problem with a random demand, re-order time is fixed and known, no lead time and production is instantaneous, shortages are allowed and they are back-logged.

5. Let X_n be the amount of grain produced in the year $(n, n+1)$ $(n = 0, 1, 2, \ldots)$. A proportion α $(0 < \alpha < 1)$ of the amount of the grain available during a year is stored for future use. If X_n $(n = 0, 1, 2, \ldots)$ are independently and identically distributed with finite variance obtain the expected value and the variance of the stock in the long run.

6. Construct the mathematical model for the following inventory problem :

   "Stock is reviewed continuously and an order of size y is placed every time the stock level reaches a certain reorder point R. The p.d.f. of demand during lead time is given as $f(x)$. p and h denote the penalty cost and holding cost per unit time. K is the set-up cost per order".

7. Demand $d(t)$ or a certain product $(0, t)$ follows a poisson process with mean λt. A store with finite capacity K is supplying the demand. Orders to replenish the stock are placed for M items per order at times at which $d(t) = kM$, $k = 1, 2, \ldots$. Ordered material is received after a fixed time lag T. Obtain the stationary distribution of the stock function.

8. The demand for a product during a fixed period T has a known p.d.f. $f(x)$ $(0 \le x < \infty)$. The inventory holding cost of the system is C_1 per unit/time unit and shortage cost is C_2 per unit/time unit. In the beginning of each period replenishments are made so as to make an on-hand inventory level of S units, after clearing the shortages (if any). Assuming that lead time is zero, obtain : (*i*) the total cost equation of the system and (ii) expression that yields optimum value of S. What are the limitations of the model. ?

   Find the optimum value of order level S and minimum cost of the system when the demand density is :

   $$f(x) = xe^{-x} \; (x \ge 0)$$
   $$= 0 \text{ otherwise}$$

9. Determine the optimal ordering policy for a single period, when (i) the demand x has an exponential distribution; (ii) the ordering cost is $K + Cq$ for q (> 0) units; (iii) holding cost is K per unit per unit time; and (iv) the penalty cost per unit not supplied is p.

10. Discuss the importance of inventory mdels. What are the objectives that should be fulfilled by an inventory control system ? Illustrate with an example.

OBJECTIVE QUESTIONS

1. If small orders are placed frequently (rather than placing large orders infrequently), then total inventory cost is
   (a) increased. (b) reduced. (c) either increased or reduced.(d) minimized.

2. If orders are placed with size determined by the *EOQ*, then the re-order costs component is
   (a) equal to the holding cost component.
   (b) greater than the holding cost component.
   (c) less than the holding cost component.
   (d) either greater than or less then the holding cost component.

3. If *EOQ* is calculated, but an order is then placed which is smaller than this, will the variable cost:
   (a) increase. (b) decrease.
   (c)either increase or decrease. (d) no change.

4. If an optimal order size (Q^*) is calculated, but is found to be of an inappropriate size, would the total cost per unit time:
   (a) rise quickly around Q^*. (b) rise slowly around Q^*. (c) fall quickly around Q^*. (d) fall slowly around Q^*.

5. Which costs can vary with order quantity?
   (a) Unit cost only. (b) Re-order cost only. (c) Holding cost only. (d) All of these.

6. If we find a minimum on a total cost curve with discounted unit cost, then the optimal order size
   (a) at this valid minimum. (b) at or to the left of this minimum.
   (c) at or to the right of this minimum. (d) anywhere.

7. If we find a valid minimum on a total cost curve with increasing reorder cost, then the optimal order size
   (a) at this valid minimum.
   (b) at or to the left of this minimum.
   (c) at or to the right of this minimum.
   (d) anywhere.
8. When compared with instantaneous replenishment, does a finite replenishment rate lead to
   (a) the same size batches.
   (b) larger batches.
   (c) smaller batches.
   (d) either larger or smaller batches.
9. If the total investment in stock is limited, will the best order quantity for each item
   (a) equal the economic order quantity.
   (b) greater than the *EOQ*.
   (c) less than the *EOQ*.
   (d) either greater or less than the *EOQ*.
10. If the unit cost rises, will optimal order quanitity
   (a) increase ?
   (b) decrease ?
   (c) either increase or decrease ?
   (d) none of the above ?

Answers

1. (c) **2.** (a) **3.** (d) **4.** (b) **5.** (d) **6.** (c) **7.** (b) **8.** (c) **9.** (c) **10.** (b).

***

22

REPLACEMENT AND RELIABILITY MODELS

22.1. INTRODUCTION : THE REPLACEMENT PROBLEM

The replacement problems are concerned with the situations that arise when some items such as men, machines, electric-light bulbs, etc. need replacement due to their decreased efficiency, failure or break-down. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

The replacement problem arises because of the following factors :

(1) *The old item has become in worse condition and work badly or require expensive maintenance.*

(2) *The old item has failed due to accident or otherwise and does not work at all, or the old item is expected to fail shortly.*

(3) *A better or more efficient design of machine or equipment has become available in the market.*

In the case of items whose efficiency go on decreasing according to their age, it requires to spend more money on account of increased operating cost, increased repair cost, increased scrap, etc. So in such cases, the replacement of an old item with new one is the only alternative to prevent such increased expenses.

Thus the problem of replacement is to decide best policy to determine an age at which the replacement is most economical instead of continuing at increased cost. The need for replacement arises in many situations so that different type of decisions may have to be taken. For example,

(*i*) We may decide whether to wait for complete failure of the item (which might cause some loss), or to replace earlier at the expense of higher cost of the item.

(*ii*) The expensive items may be considered individually to decide whether we should replace now or, if not, when it should be reconsidered for replacement.

(*iii*) It may be decided whether we should replace by the same type of item or by different type (latest model) of item.

The problem of replacement is encountered in the case of both men and machines. Using probability it is possible to estimate the chance of death (or failure) at various ages.

The main objective of replacement is to direct the organization for maximizing its profit (or minimizing the cost).

Q. 1. What is replacement problem ? Describe some important replacement situations. **[Meerut (M.Sc. Maths) 90]**
2. Explain replacement situations giving an example for each of them. **[VTU (BE Mech.) 2003, 2002]**
3. What are the situations which makes the replacement of items necessary ? **[Meerut (M.Sc. Maths) 2000]**

22.2. FAILURE MECHANISM OF ITEMS

The term 'failure' has a wider meaning in *business* than what it has in our daily life. There are *two* kinds of failure.

(1) Gradual Failure. The mechanism under this category is progressive. That is, as the life of an item increases, its efficiency deteriorates, causing :

(*i*) increased expenditure for operating costs, (*ii*) decreased productivity of the equipments,

(*iii*) decrease in the value of the equipment, *i.e.*, the resale of saving value decreases.

For example, mechanical items like *pistons, bearings, rings* etc. Another example is *'Automobile tyres'*.

(2) Sudden Failure. This type of failure is applicable to those items that do not deteriorate markedly with service but which ultimately fail after some period of using. The period between installation and failure is not constant for any particular type of equipment but will follow some frequency distribution which may be progressive, retrogressive or random in nature.

(*i*) **Progressive failure :** Under this mechanism, probability of failure increases with the increase in the life of an item. For example, *electric light bulbs, automobile tubes,* etc.

(*ii*) **Retrogressive failure :** Certain items have more probability of failure in the beginning of their life, and as the time passes the chances of failure become less. That is, the ability of the unit to survive in the initial period of life increases its expected life. Industrial equipments with this type of distribution of life span is examplified by air craft engines.

(*iii*) **Random failure :** Under this failure, constant probability of failure is associated with items that fail from random causes such as *physical shocks,* not related to age. In such a case, virtually all items fail before aging has any effect. For example, vacuum tubes in air-borne equipment have been shown to fail at a rate independent of the age of the tube.

The replacement situations may be placed into *four* categories :

(1) Replacement of capital equipment that becomes worse with time, *e.g. machines tools, buses in a transport organization, planes,* etc.

(2) Group replacement of items that fail completely, *e.g., light bulbs, radiô tubes,* etc.

(3) Problems of mortality and staffing.

(4) Miscellaneous Problems.

Q. 1. Explain with examples the failure mechanism of items.

2. Explain the difference between age replacement and preventive maintenance.

3. What is the advantage of preventive replacement over routine replacement ? **[Tamilnadue (BE) 97]**

4. Name the type of models.

I–Replacement of Items that Deteriorate

22.3. COSTS TO BE CONSIDERED

In general, the costs to be included in considering replacement decisions are all those costs that depend upon the choice or age of machine. In some special problems, certain costs need not be included in the calculations. For example, in considering the optimum decision of replacement for a particular machine, the costs that do not change with the age of the machine need not be considered.

22.4. WHEN THE REPLACEMENT IS JUSTIFIED ?

This question can easily be answered by considering a case of truck owner whose problem is to find the 'best' time at which he should replace the old truck by new one. The truck owner wants to transport goods as cheaply as possible. The associated costs are :

(*i*) the running costs, and (*ii*) the capital costs of purchasing a truck.

These associated costs can be expressed as average cost per month. Now the truck owner will observe that the average monthly cost will go on decreasing, longer the replacement is postponed. However, there will come an age at which the rate of increase of running costs more than compensates the saving in average capital costs. Thus, at this age the replacement is justified.

22.5. REPLACEMENT POLICY FOR ITEMS WHOSE MAINTENANCE COST INCREASES WITH TIME, AND MONEY VALUE IS CONSTANT

Theorem 22.1. *The cost of maintenance of a machine is given as a function increasing with time and its scrap value is constant.*

(a) If time is measured continuously, then the average annual cost will be minimized by replacing the machine when the average cost todate becomes equal to the current maintenance cost.

(b) If time is measured in discrete units, then the average annual cost will be minimized by replacing the machine when the next periods maintenance cost becomes greater than the current average cost.

Proof. (*a*)—When time '*t*' is a continuous variable.

Let R_t = maintenance cost at time t, C = the capital cost of the item, S = the scrap value of the item.

Obviously, annual cost of the item at any time $t = R_t + C - S$.

Since the maintenance cost incurred during 'n' years becomes $= \int_0^n R_t dt$, the total cost incurred on the item will become :

$$P(n) = \int_0^n R_t dt + C - S.$$

Hence average total cost is given by $$F(n) = \frac{P(n)}{n} = \frac{1}{n}\int_0^n R_t\, dt + \frac{C-S}{n} \qquad ...(22.1)$$

Now, we have to find such time n for which $F(n)$ is minimum. Therefore, differentiating $F(n)$ w.r.t. 'n',

$$\frac{dF(n)}{dn} = \frac{1}{n} R_n + \left(-\frac{1}{n^2}\right)\int_0^n R_t\, dt - \frac{C-S}{n^2} = 0, \text{ for minimum of } F(n), \qquad ...(22.2)$$

which gives $$R_n = \frac{1}{n}\int_0^n R_t\, dt + \frac{C-S}{n} = \frac{P(n)}{n}, \text{ by virtue of equation (22.1)} \qquad ...(22.3)$$

Hence, maintenance cost at time n = average cost in time n.

(*b*) **When time '*t*' is a discrete variable.**

Since the time is considered in discrete units, the cost equation (22.1) can be written as

$$F(n) = \frac{P(n)}{n} = \sum_{t=1}^{n} \frac{R_t}{n} + \frac{C-S}{n}, \qquad ...(22.4)$$

By using finite differences, $F(n)$ will be minimum if the following relationship is satisfied. :

$$\Delta F(n-1) < 0 < \Delta F(n) \qquad ...(22.5)$$

Now, differencing (22.4) under the summation sign by definition of first difference,

$$\begin{aligned}\Delta F(n) &= F(n+1) - F(n)\\ &= \left[\sum_{t=1}^{n+1} \frac{R_t}{n+1} + \frac{C-S}{n+1}\right] - \left[\sum_{t=1}^{n} \frac{R_t}{n} + \frac{C-S}{n}\right]\\ &= \left(\frac{R_{n+1}}{n+1} + \sum_{t=1}^{n} \frac{R_t}{n+1}\right) - \sum_{t=1}^{n} \frac{R_t}{n} + (C-S)\left[\frac{1}{n+1} - \frac{1}{n}\right]\\ &= \frac{R_{n+1}}{n+1} + \sum_{t=1}^{n} R_t\left(\frac{1}{n+1} - \frac{1}{n}\right) + (C-S)\left[\frac{1}{n+1} - \frac{1}{n}\right] \qquad ...(22.6)\\ &= \frac{R_{n+1}}{n+1} - \sum_{t=1}^{n} \frac{R_t}{n(n+1)} - \frac{C-S}{n(n+1)}.\end{aligned}$$

Since $\Delta F(n) > 0$ for minimum of $F(n)$, so

$$\frac{R_{n+1}}{n+1} > \sum_{t=1}^{n} \frac{R_t}{n(n+1)} + \frac{C-S}{n(n+1)} \quad \text{or} \quad R_{n+1} > \sum_{t=1}^{n} \frac{R_t}{n} + \frac{C-S}{n}$$

or $$R_{n+1} > P(n)/n, \text{ by virtue of equation (22.4).} \qquad ...(22.7)$$

Similarly, it can be shown that $R_n < P(n)/n$, by virtue of $\Delta F(n-1) < 0$

Hence $$R_{n+1} > (P(n)/n) > R_n.$$

This completes the proof.

Q. 1. Describe the problem of replacement of items whose maintenance costs increase with time. You may assume that the value of money remains constant. Hence establish the following rule for replacement. Do not replace if the next periods cost is less than the weighted average of previous costs. **[Agra 99]**

2. The cost of maintenance of a machine is given as a function increasing with time and its scrap value is constant. Show that the average annual cost will be minimized by replacing the machine when the average cost to date becomes equal to the current maintenance cost. **[Meerut 2002; Raj. Univ. (M. Phil) 93]**

3. Discuss the policy of replacement of items whose maintenance cost increases with time but the value of money remains constant during the period. **[Garhwal M.Sc. (Stat.) 94; Raj. Univ. (M. Phil) 91]**

4. Derive the expression for the average annual cost of an item over a period of n years, when the **money** value remains constant. **[Tamilnadue 97]**

5. Write a brief note on 'Replacement policy of items which deteriorate with time'. **[VTU 2002]**

22.6. ILLUSTRATIVE EXAMPLES

Example 1. *The cost of a machine is Rs. 6100 and its scrap value is only Rs. 100. The maintenance costs are found from experience to be :*

Year	:	1	2	3	4	5	6	7	8
Maintenance cost in Rs.	:	100	250	400	600	900	1250	1600	2000

When should machine be replaced ? **[JNTU 99; Raj. Univ. (M.Phil) 93; Meerut (Stat.) 90]**

Solution. First, find an average cost per year during the life of the machine as follows :

Total cost in first year = Maintenance cost in first year + loss in purchase price

= 100 + (6100 − 100) = Rs. 6100

∴ Average cost in first year = Rs. 6100.

Total cost upto two years = Maintenance cost upto two years + loss in purchase price

= (100 + 250) + 6000 = Rs. 6350.

∴ Average cost per year during first two years = Rs. 3175.

In a similar fashion, average cost per year during first three years = 6750/3 = Rs. 2250.00,
average cost per year during first four years = 7350/4 = Rs. 1837.50,
average cost per year during first five years = 8250/5 = Rs. 1650.00,
average cost per year during first six years = 9500/6 = Rs. 1583.33,
average cost per year during first seven years = 11100/7 = Rs. 1585.71 (*Note*).

These computations may be summarized in the following tabular form.

Table 22.1.

Replace at the end of year (n)	Maintanence cost (R_n)	Total maintenance cost (ΣR_n)	Difference between Price and Resale price $(C - S)$	Total cost $P(n)$	Average cost $\frac{P(n)}{n}$
(1)	(2)	(3)	(4)	(5) = (3) + (4)	(6)
1	100	100	6000	6100	6100
2	250	350	6000	6350	3175
3	400	750	6000	6750	2250
4	600	1350	6000	7350	1837
5	900	2250	6000	8250	1650
→6	1250	3500	6000	9500	**1583**
7	**1600**	5100	6000	11100	1586
8	2000	7100	6000	13100	1638

Here it is observed that the maintenance cost in the 7th year becomes greater than the average cost for 6 years [*i.e.* $R_7 > P(6)/6$]. Hence the machine should be replaced at the end of 6th year.

Alternatively, last column of above table shows that the average cost starts increasing in the 7th year, so the machine should be replaced before the beginning of 7th year, *i.e.* at the end of 6th year.

Example 2. *A machine owner finds from his past records that the costs per year of maintaining a machine whose purchase price is Rs. 6000 are as given below :*

Year	:	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	1000	1200	1400	1800	2300	2800	3400	4000
Resale Price	:	3000	1500	750	375	200	200	200	200

Determine at what age is a replacement due ?

[Meerut 2002; JNTU 2002 98; Agra 99; I.A.S. (Main) 93; Rohilkhand 93]

Solution. *Table* 22.2 shows the average cost per year during the life of the machine. Here C = Rs. 6000.

Table 22.2

Replace at the end of year (n)	Maintenance cost (R_n)	Total maintenance cost (ΣR_n)	Difference between Price and Resale price ($C - S_n$)	Total cost $P(n)$	Average cost $\frac{P(n)}{n}$
(1)	(2)	(3)	(4)	(5) = (3) + (4)	(6)
	Rs.	Rs.	Rs.	Rs.	Rs.
1	1000	1,000	3,000	4,000	4,000
2	1200	2,200	4,500	6,700	3,350
3	1400	3,600	5,250	8,850	2,950
4	1800	5,400	5,625	11,025	2,756
→5	2300	7,700	5,800	13,500	**2,700**
6	**2800**	10,500	5,800	16,300	2,717

As in ***Example 1***, the machine should be replaced at the end of fifth year, because the maintenance cost in the 6th year becomes greater than the average cost for 5 years.

Example 3. *The machine owner has three machines of purchase price Rs. 6000 each and cost per year of maintaining each machine is same as in* ***Example 2.*** *Two of these machines are two years old and the third is one year old. He is considering a new machine of purchase price Rs. 8000 with 50% more capacity than one of the old ones. The estimates of maintaining cost and resale price for new machine are as given below :*

Year	:	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	1200	1500	1800	2400	3100	4000	5000	6100
Resale price (Rs.)	:	4000	2000	1000	500	300	300	300	300

Assuming that the loss of flexibility due to fewer machines is of no importance, and he continues to have sufficient work for three of the old machines, what should his policy be ?

Solution. First compute the average yearly costs during the life of a new machine of larger capacity as computed for old machine in ***Example 2.***

Observation : 4000 > 3540 *i.e.* $R_6 > P(5)/5$. ***Discision*** : Replace at the end of 5th year.

Now, decide whether it is economically justified to replace smaller machine by new larger machine.

Since the new larger machine has 50% more capacity than that of smaller one, three smaller machines will be equivalent to two larger machines. Consequently, the lowest average cost of **Rs. 3540** (see *Table* 22.3) for new larger machine is equivalent to (Rs. 3540 × 2/3) *i.e.* Rs. 2360 per pay-load of the smaller machines. Since the amount of **Rs. 2360** is less than the minimum average annual cost **Rs. 2700** (see *Table* 22.2) for one of the old smaller machines, hence the latter will be replaced by a new larger machine.

Table 22.3.

Replace at the end of year (n)	Maintenance cost Rs. (R_n)	Total maintenance cost Rs. (ΣR_n)	Difference between Price and Resale price Rs. ($C - S$)	Total cost $P(n)$	Average cost Rs. $\frac{P(n)}{n}$
(1)	(2)	(3)	(4)	(5) = (3) + (4)	(6)
1	1200	1,200	4,000	5,200	5,200
2	1500	2,700	6,000	8,700	4,350
3	1800	4,500	7,000	11,500	3,833
4	2400	6,900	7,500	14,400	3,600
→5	3100	10,000	7,700	17,700	**3,540**
6	**4000**	14,000	7,700	21,700	3,616

Now, decide when the new large machine should be purchased. Assume that for uniformity, the replacement will involve two new machines and all the three old machines. The new machine should be purchased when the cost for the next year of running the three old machines exceeds the average yearly cost for two new type of machines.

It is seen from *Table* 22.2 that total yearly cost for one smaller machine is as follows :

Total cost during first year = Rs. 4000,

Total cost during second year = Rs. (6700 − 4000) = Rs. 2700,

Total cost during third year = Rs. (8850 − 6700) = Rs. 2150,

and similarly, Rs. 2175, Rs. 2475, Rs. 2800 during the 4th, 5th and 6th years respectively.

Hence total cost next year for smaller machine aged two years and one smaller machine aged one year becomes = 2 × 2150 + 2700 = Rs. 7000. Similarly,

total cost during 2nd year = 2 × 2175 + 2150 = Rs. 6500,
total cost during 3rd year = 2 × 2475 + 2175 = Rs. 7125,
total cost during 4th year = 2 × 2800 + 2475 = Rs. 8075.

But, the minimum average cost for two larger machines will be = 2 × 3540 = **Rs. 7080.** (*Note*)

It has been observed that the cost (Rs. 6500) for the old machines will not exceed the cost (Rs. 7080) for the larger new machines until the 3rd year. Hence all the three small machines should be replaced after two years before any of them reaches the normal replacement age of 5 years (as seen from *Table* 22.2)

Example 4. *(a) Machine A costs Rs. 9000. Annual operating cost is Rs. 200 for the first year, and then increase by Rs. 2000 every year, i.e. in the fourth year operating cost becomes Rs. 6200. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine ? (Assume that the machine has no resale value when replaced and that future costs are not discounted).* **[Meerut M.Sc. (Math.) BP-96]**

(b) Machine B costs Rs. 10,000. Annual operating cost is Rs. 400 for the first year, and then increased by Rs. 800 every year. You have own a machine of type A which is one year old. Should you replace it with B, and if so when ? **[Agra 98]**

(c) Suppose you are just ready to replace machine A with another machine of the same type, when you hear that machine B will become available in a year. What should you do ?

Solution. *(a)* For machine A, the average cost per year can be computed from *Table* 22.4.

Table 22.4. (For machine A)

Replace at the end of year (n)	Running cost (R_n)	Total running cost (ΣR_n)	Loss due to resaling ($C-S$)	Total cost $P(n)$	Average cost per year $P(n)/n$
(1)	(2)	(3)	(4)	(5) = (3) + (4)	(6)
	(Rs.)	(Rs.)	(Rs.)	(Rs.)	(Rs.)
1	200	200	9,000	9,200	9,200
2	2200	2,400	9,000	11,400	5,700
→3	4200	6,600	9,000	15,600	5,200
4	6200	12,800	9,000	21,800	5,450

Thus, the machine *A* should be replaced at the end of third year and average yearly cost of owning and operating the machine is Rs. 5200 at the optimum time (3 years) of replacement.

(*b*) In a similar fashion, prepare *Table* 22.5 for machine *B*.

Table 22.5 (For machine B)

Replace at the end of year (n)	Running cost (R_n)	Total running cost (ΣR_n)	Loss due to resaling ($C-S$)	Total cost $P(n)$	Average cost per year $P(n)/n$
(1)	(2)	(3)	(4)	(5) = (3) + (4)	(6)
	Rs.	Rs.	Rs.	Rs.	Rs.
1	400	400	10,000	10,400	10,400
2	1200	1,600	10,000	11,600	5,800
3	2000	3,600	10,000	13,600	4,533
4	2800	6,400	10,000	16,400	4,100
→5	3600	10,000	10,000	20,000	4,000
6	4400	14,400	10,000	24,400	4,066

Since the lowest average cost is Rs. 4000 for machine *B* and is less than the lowest average cost of Rs. 5200 for machine *A*, so *A* can be replaced by machine *B*.

Now decide when the machine *B* should be purchased. The machine *B* should be purchased when *the cost for next year of running the machine A exceeds the average yearly cost for machine B.*

Find the total yearly cost for machine *A* as follows :

for 1st year = 11400 − 9200 = 2200 < 4000, for 2nd year = 15600 − 11400 = 4200 > 4000,
for 3rd year = 21800 − 15000 = 6200 > 4000.

Thus, it has been observed that the cost (Rs. 2200) for one year old machine A will not exceed the lowest average cost (Rs. 4000) for B until the second year hence. Therefore, A should be replaced after one year from now before it reaches the normal replacement age of three years.

(c) Instead of replacing machine A with another machine of the same type, purchase machine B which is available after a year. The reason is that machine A can be replaced with B after one year from now as seen above in *(b)*.

Example 5. *Fleet cars have increased their costs as they continue in service due to increased direct operating cost (gas and oil) and increased maintenance (repairs, tyres, batteries, etc.). The initial cost is Rs. 3,500, and the trade-in value drops as time passes until it reaches a constant value of Rs. 500.*

Given the cost of operating, maintaining and the trade-in value, determine the proper length of service before cars should be replaced.

Year of service	:	1	2	3	4	5
Year end trade-in value	:	1,900	1,050	600	500	500
Annual operating cost	:	1,500	1,800	2,100	2,400	2,700
Annual maintaining cost	:	300	400	600	800	1,000

Solution. Compute the following table :

Table 22.6.

Year (n)	Operating + Maintenance cost (R_n)	Total cost (ΣR_n)	Trade in value (S_n)	Difference $(C - S_n)$	Total cost $P(n)$	Average cost $P(n)/n$
1	1,800	1,800	1,900	1,600	3,400	3,400
2	2,200	4,000	1,050	2,450	6,450	3,225
→3	2,700	6,700	600	2,900	9,600	3,200
4	3,200	9,900	500	3,000	12,900	3,225
5	3,700	13,600	500	3,000	16,600	3,320

This table shows that the car should be replaced at the end of every third year.

EXAMINATION PROBLEMS

1. A firm is considering replacement of a machine whose cost price is Rs. 12,200; and the scrap value is Rs. 200. The maintenance costs are found from experience to be as follows :

Year	:	1	2	3	4	5	6	7	8
Mainte. Cost (Rs.)	:	200	500	800	1200	1800	2500	3200	4000

When should the machine be replaced ?
[**Ans.** Replace at the end of 6th year.] **[JNTU 2000]**

2. The following table gives the running costs per year and resale price of a certain equipment whose purchase price is Rs. 5000.

Year	:	1	2	3	4	5	6	7	8
Running costs (Rs.)	:	1500	1600	1800	2100	2500	2900	3400	4000
Resale value (Rs.)	:	3500	2500	1700	1200	800	500	500	500

At what year is the replacement due ?
[**Ans.** At the end of 5th year.] **[Meerut (Stat.) 95, (Math) 93]**

3. A truck owner finds from his past records that the maintenance costs per year of a truck whose purchase price is Rs. 8000, are given below :

Year	:	1	2	3	4	5	6	7	8
Mainte. Cost (Rs)	:	1000	1300	1700	2200	2900	3800	4800	6000
Resale Price (Rs.)	:	4000	2000	1200	600	500	400	400	400

Determine at what time it is profitable to replace the truck.
[**Ans.** Replace at the end of 5th year.]

4. A new tempo costs Rs. 8000 and may be sold at the end of any year at the following prices :

Year (end)	:	1	2	3	4	5	6
Selling Price (Rs.)	:	5,000	3,300	2,000	1,100	600	100 (at present value)
Annual operating cost (Rs.)	:	1,000	1,200	1,500	2,000	3,000	5,000 (at present value)

It is not only possible to sell tempo after use but also to buy a second hand tempo. It may be cheaper to do so than to replace with a new tempo.

Age of tempo	:	0	1	2	3	4	5
Purchase Price (Rs.)	:	8,000	5,800	4,000	2,600	1,600	1,000 (at present value)

What is the age to buy and to sell the tempo and to minimize average annual cost ?

[**Hint.** Prepare tables for the new tempo and second hand tempo. The lowest average annual cost (Rs. 3150) can be achieved by replacing the tempo after 4 years. Similarly, the lowest average cost of second hand tempo is obtained (Rs. 3,025) after 4 years. Clearly, it will be cheaper if the second hand tempo is replaced after 4 years instead of new one saving thereby Rs. (3,150 – 3,025) = Rs. 125.]

[**Note**. Since for new tempo $R_5 < P(4)/4$ (*i.e.* $3000 < 3150$), we do not replace after 4 years. But, $R_6 > P(5)/5$ (*i.e.* $5000 > 3220$), therefore replace at the end of 5 years.]

5. A taxi owner estimates from his past records that the cost per year for operating a taxi whose purchase price when new is Rs. 60,000 are as given below:

Age	:	1	2	3	4	5
Operating cost	:	10,000	12,000	15,000	18,000	20,000

After 5 years, the operating cost = $6000k$, where $k = 6, 7, 8, 9, 10$ (k denoting age in years).

If the resale value decreases by 10% of purchase price each year, what is the best replacement policy ? Cost of money is zero.

[**Ans.** Since the minimum average cost is during the first year, the taxi should be replaced at the end of every year.]

6. After the present machine has done 5 years service, an offer is made of second hand equivalent machine costing Rs. 7,500. This alternative equipment is expected to need Rs. 400 as spare cost in first year, which is likely to rise by Rs. 500 per year. Scrap value is expected to be zero. Should the offer be taken.

7. Madras Cola Inc. uses a bottling machine that costs Rs. 50,000 when new. Table below gives the expected operating costs per year, the annual expected production per year and the salvage value of the machine. The wholesale price for a bottle of drink is Re. 1.00.

Data Associated with Age of Bottling Machine

Age	:	1	2	3	4	5
Operating costs (Rs.)	:	7,000	8,000	10,000	14,000	20,000
Production (Bottles)	:	2,08,000	2,08,000	2,00,000	1,90,000	1,75,000
Salvage value (Rs.)	:	30,000	19,000	15,000	12,000	10,000

When should machine be replaced ?
[**Ans.** Replace at the end of 4th year.]

8. A new three-wheeler auto costs Rs. 20,000 and may be resale at the end of any year at the following prices :

Year (end)	:	1	2	3	4	5	6
Selling Price (Rs.)	:	15,000	13,000	10,000	8,000	6,000	5,000 *(at present value)*

The corresponding annual operating costs are :

Year (end)	:	1	2	3	4	5	6
Cost/Year (Rs.)	:	5,000	6,000	7,500	8,000	8,500	9,500 *(at present value)*

It is not only possible to sell the auto after use but also to buy a second-hand auto of different make. It may be cheaper to do so than to replace with a new auto.
The purchase price of the auto of this make is as follows :

Age of Auto	:	0	1	2	3	4	5
Purchase Price (Rs.)	:	20,000	16,000	13,000	10,000	8,000	6,000 *(at present value)*

What is the age to buy and to sell auto to minimize average annual cost ?
[**Ans.** End of 2nd year]

9. A fleet owner finds from his past records that the costs per year of running a vehicle whose purchase price is Rs. 50,000 are as under :

Year	:	1	2	3	4	5	6	7
Running cost (Rs.)	:	5,000	6,000	7,000	9,000	21,500	16,000	18,000
Resale value (Rs.)	:	30,000	15,000	7,500	3,750	2,000	2,000	2,000

Thereafter, running cost increases by Rs. 2000, but resale value remains constant at Rs. 2,000. At what age is a replacement due ?

10. A truck owner from his past experience estimated that the maintenance cost per year of a truck whose purchase price is Rs. 1,50,000 and the resale value of truck will be as follows :

Year	:	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	10,000	50,000	20,000	25,000	30,000	40,000	45,000	50,000
Resale value (Rs.)	:	1,30,000	1,20,000	1,15,000	1,05,000	90,000	75,000	60,000	50,000

Determine at which time it is profitable to replace the truck. **[VTU 2003]**

[**Ans.** Though the minimum value of the average cost occurs at the end of first year, but it is impractical to replace the truck every year. So, next time the minimum average cost occurs at the end of 4th year. Hence it is profitable to replace the truck after every 4 years.
Note. Here the condition for replacement policy maintenance cost increases with time violating in the 3rd year]

11. A machine costs Rs. 10,000. Operating costs are Rs. 50 per year for the first five years. In the sixth and succeeding years operating costs increase by Rs. 100 per year. Find optimum length of time to hold the machine before replacing it.

[**Ans.** End of 15 years.]

12. For a machine the following data are availabe.

Year	Cost of spares (Rs.)	Salary of Maintenance Staff (Rs.)	Loss due to break-downs (Rs.)	Resale value (Rs.)
0	—	—	—	20000
1	100	1600	500	14000
2	500	1600	700	12000
3	700	1600	500	10000
4	900	2000	1000	6000
5	1300	2400	1500	3000
6	1600	2400	1600	800

Determine the optimum policy for replacement of above machine. **[I.A.S. (Math.) 95]**

13. Find the optimal replacement age of an item from the following data, given the maintenance cost M_r in the r-th year and the total cost of the item T_r at the end of r-th year :

r :	1	2	3	4	5
M_r :	2	22	42	62	82
T_r :	92	114	156	218	300

[Tamilnadue (BE) 97]

14. Machine A costs Rs. 3,600. Annual operating costs are Rs. 40 for the first year and then increase by Rs. 360 every year. Assuming that Machine A has no resale value, determine the best replacement age. Another machine B, which is similar to machine A, costs Rs. 4,000. Annual running costs are Rs. 200 for the first year and then increase by Rs. 200 every year. It has resale value of Rs. 1,500, Rs. 1,000 and Rs. 500 at the and of Ist, 2nd and 3rd year respectively. It has no resale value during 4th year and onwards.

15. A machine has initial investment of Rs. 30,000 and its salvage value at the end of 'i' years of its use is estimated as Rs. 30,000/(i + 1). The annual operating and maintenance cost in the first year is Rs. 15,000 and increases by Rs. 1000 in each subsequent years for the first five years and increases by Rs. 5000 in each year thereafter. Replacement policy to be planned over a period of seven years. During this period cost of capital may be taken as 10% per year. State the problem for optimal replacement.

16. A mechine has been purchased at a cost of Rs. 1,60,000. The value of the machine is depreciated in the first three years by Rs. 20,000 each year and Rs. 16,000 per year thereafter. Its maintenance and operating cost for the first area years are Rs. 16,000, Rs. 18,000 and Rs. 20,000 in that order and increase by Rs. 4000 every year. Assuming an interest rate of 10%, find the economic life of the machine. **[JNTU (B. Tech.) 2003]**

22.7. MONEY VALUE, PRESENT WORTH FACTOR (*PWF*), AND DISCOUNT RATE

Money value. Since money has a value over time, we often speak : *'money is worth 10% per year'*. This can be explained in the following two alternative ways :

(*i*) *In one way,* spending Rs. 100 today would be equivalent to spending Rs. 110 in a year's time. In other words, if we plan to spend Rs. 110 after a year from now, we could equivalently spend Rs. 100 today which would be of worth Rs. 110 next year.

(*ii*) *Alternatively,* if we borrow Rs. 100 at the rate of interest 10% per year and spend this amount today, then we have to pay Rs. 110 after one year.

Thus we conclude that Rs. 100 today will be equivalent to Rs. 110 after a year from now. Consequently, one rupee after a year from now is equivalent to $(1.1)^{-1}$ rupee today.

Present worth factor (*Pwf*). As we have just seen above, one rupee a year from now is equivalent to $(1.1)^{-1}$ rupee today at the interest rate 10% per year. One rupee spent *two* years from now is equivalent to $(1.1)^{-2}$ today. Similarly, we can say one-rupee spent n years from now is equivalent to $(1.1)^{-n}$ today. This quantity $(1.1)^{-n}$ is called the ***Present worth factor (Pwf)*** or ***present value*** of one rupee spent n years from now.

More generally, if r is the interest rate, then $(1 + r)^{-n}$ is called the ***present worth factor (Pwf)*** or ***present value*** of one rupee spent in n years time from now onwards. The expression $(1 + r)^{-n}$ is known as the *Compound amount factor* (*Caf*) of one rupee spent in n years duration.

Discount rate (*Depreciation value*). The *present worth factor* of unit amount to be spent after one year is given by $v = (1 + r)^{-1}$, where r is the interest rate. Then, v is called ***discount rate*** (technically known as the ***depreci- ation value***). Following example will explain the effect of considering money value with prescribed rate of interest.

Q. Explain money value, present value, and discount rate. **[Garhwal M.Sc. (Stat.) 96]**

Example 6. *The cost pattern for two machines A and B, when money value is not considered, is given in Table* 22.7;

Table 22.7.

Year	Cost at the beginning of year (in Rs.)	
	Machine A	Machine B
1	900	1400
2	600	100
3	700	700

Find the cost pattern for each machine when money is worth 10% per year, and hence find which machine is less costly.

Solution. The total outlay for three years for machine $A = 900 + 600 + 700 =$ Rs. 2200, and also for machine $B = 1400 + 100 + 700 =$ Rs. 2200.

Here we observe that the total outlay for either machine is same for three years when the money value is not taken into account. Hence both the machines will appear to be equally good in this case.

Now consider the money value at the rate of 10% per year, the discount cost pattern for each machine for three years is shown in *Table* 22.8 :

Table 22.8.

Year	Discounted cost (10% rate) in Rs.	
	Machine A	Machine B
1	900.00	1400.00
2	$600 \times 100/110 = 545.45$	$100 \times 100/110 = 90.90$
3	$700 \times (100/110)^2 = 578.52$	$700 \times (100/110)^2 = 578.52$
Total outlay	Rs. 2023.97	Rs. 2069.43

The data show that the total outlay for machine A is actually Rs. 45.46 less than that of machine B. Hence machine A will be preferred.

Example 7. *Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every six years. The yearly costs of both the machines are given as under :*

Year :	1	2	3	4	5	6
Machine A :	1,000	200	400	1,000	200	400
Machine B :	1,700	100	200	300	400	500

Determine which machine should be purchased.

Solution. Present worth factor is given by $v = \frac{100}{100+10} = \frac{10}{11}$.

$\therefore$ Total discount cost (present worth) of A for 3 years $= \text{Rs.}\left[1,000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2\right]$

$=$ Rs. 1,512 (nearly).

Again the total discount cost of B for six years

$$= \text{Rs.}\left[1,700 + 100 \times \frac{10}{11} + 200 \times \left(\frac{10}{11}\right)^2 + 300 \times \left(\frac{10}{11}\right)^3 + 400 \times \left(\frac{10}{11}\right)^4 + 500 \times \left(\frac{10}{11}\right)^5\right]$$

$=$ Rs. 2765.

Average yearly cost of $A = 1{,}512/3 =$ Rs.504

and average yearly cost of $B = 2{,}765/6 =$ Rs. 461.

Although, this shows the apparent advantage with B, but the comparison is unfair because the periods of consideration are different.

So, if we consider 6 years period for machine A also, then the total discount of A will be

$$= 1,000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2 + 1,000 \times \left(\frac{10}{11}\right)^3 + 200 \times \left(\frac{10}{11}\right)^4 + 400 \times \left(\frac{10}{11}\right)^5 = \text{Rs. } 2647,$$

which is Rs. 118 less costlier than machine B over the same period.

Hence machine A should be purchased.

Example 8. *A manual stamper currently valued at Rs. 1,000 is expected to last 2 years and costs Rs. 4,000 per year to operate. An automatic stamper which can be purchased for Rs. 3,000 will last 4 years and can be operated at an annual cost of Rs. 3,000. If money carries the rate of interest 10% per annum, determine which stamper should be purchased.*

Solution. The present worth factor is given by, $v = \frac{100}{100+10} = 0.9091$

The two given stampers have different expected lives. So, we shall consider a span of four years during which we have to purchase either two manual stampers (the second one being purchased in the third year) or one automatic stamper.

The present worth of investments on the two manual stampers used in 4 years is :

$$= 1000(1 + v^2) + 4000\,(1 + v + v^2 + v^3)$$
$$= 1000[1 + (0.9091)^2] + 4000 + 4{,}000[(0.9091) + (0.9091)^2 + (0.9091)^3]$$
$$= 1000[1 + 0.8264] + 4000 + 4000[0.9091 + 0.8264 + 0.7513]$$
$$= 1{,}826 + 13947 = \text{Rs. } 15773 \text{ (approx.).}$$

Also, the present worth of investments on the automatic stamper for the next four years

$$= 3{,}000 + 3{,}000\,(1 + v + v^2 + v^3) = 3{,}000 + 10460 = \text{Rs.}13{,}460.$$

Since the present worth of future costs for the automatic stamper is less than that of the manual stamper, it will be more profitable to purchase an automatic stamper.

Example 9. *A pipeline is due for repairs. It will cost Rs. 10,000 and lasts for 3 years. Alternatively, a new pipeline can be laid at a cost of Rs. 30,000 and lasts for 10 years. Assuming cost of capital to be 10% and ignoring salvage value, which alternative should be chosen.*

Solution. Consider two types of pipeline for infinite replacement cycles of 10 years for the new pipeline, and 3 years for the existing pipeline.

The present worth factor is given by, $v = \frac{100}{100+10} = \frac{10}{11} = 0.9091$

Let D_n denote the discounted value of all future costs associated with a policy of replacing the equipment after n years. Then, if we designate the initial outlay by C,

$$D_n = C + v^n C + v^{2n} C + \ldots = C\,[1 + v^n + v^{2n} + \ldots\ldots] = C/(1 - v^n)$$

Now substituting the values of C, v and n for two types of pipelines; the discounted value for the existing pipeline is given by

$$D_3 = \frac{10{,}000}{1-(0.9091)^3} = \text{Rs. } 40021.$$

and for the new pipeline

$$D_{10} = \frac{30{,}000}{1-(0.9091)^{10}} = \frac{30{,}000}{1-0.3855} = \text{Rs. } 48820.$$

Since $D_3 < D_{10}$, the existing pipeline should be continued.

Alternatively, the comparison may be made over $3 \times 10 = 30$ years.

Example 10. *A person is considering to purchase a machine for his own factory. Relevant data about alternative machines are as follows :*

		Machine A	Machine B	Machine C
Present Investment (Rs.)	:	10,000	12,000	15,000
Total annual cost (Rs.)	:	2,000	1,500	1,200
Life (years)	:	10	10	10
Salvage value	:	500	1,000	1,200

As an adviser to be buyer, you have been asked to select the best machine, considering 12% normal rate of return.

You are given that—

(i) Single payment present worth factor (Pwf) @ 12% for 10 years = 0.322.

(ii) Annual series present worth factor (Pwf) @ 12% for 10 years = 5.650. **[I.A.S (Main) 94]**

Solution. The present value of total cost of each of the three machines for a period of 10 years is given as below :

Machine	Present Investment	Present value of total annual cost	Present value of salvage value	Net cost Rs.
	(1)	(2)	(3)	(1) + (2) – (3)
A	10,000	2,000 × 5.65 = 11,300	500 × 0.322 = 161.00	21139.00
B	12,000	1,500 × 5.65 = 8,475	1000 × 0.322 = 322.00	20153.00
C	15,000	1,200 × 5.65 = 6,780	1200 × 0.322 = 386.40	21393.60

From above table, we conclude that the present value of table cost for machine *B* is the least and hence machine *B* should be purchased.

22.8. REPLACEMENT POLICY WHEN MAINTENANCE COST INCREASES WITH TIME AND MONEY VALUE CHANGES WITH CONSTANT RATE

As already explained in the preceding section, the money value can be interpreted in two different ways. Accordingly, the optimal replacement policy can be determined by the following two methods :

(*i*) The maintenance cost increases with time and the money value decreases with constant rate *i.e.* depreciation value is given.

(*ii*) The amount to be spent is borrowed at a given rate of interest under the condition of repaying it in pre-decided number of instalments.

Theorem 22.2. *The maintenance cost increases with time and the money value decreases with constant rate i.e. depreciation value is given. Then replacement policy will be—*

(*i*) *Replace if the next period's cost is greater than the weighted average of previous costs.*

(*ii*) *Do not replace if the next period's cost is less than the weighted average of previous costs.*

Proof. ***First Method.*** Suppose that the item (which may be a machine or equipment etc.) is available for use over a series of time periods of equal intervals (say, one year).

Let C = purchase price of the item to be replaced

R_i = running (or maintenance) cost incurred at the beginning of year

r = rate of interest

v = $1/(1+r)$ is the present worth of a rupee to be spent a year hence.

The process can be divided into two major steps :

Step 1. ***To find the present worth of total expenditure.***

Let the item be replaced at the end of every nth year. The yearwise present worth of expenditure on the item in the successive cycles of n years can be calculated as follows :

Year	1	2	...	n	$n+1$	$n+2$	.....	2n	$2n+1$	...
Present worth	$C+R_1$	R_2v	...	R_nv^{n-1}	$(C+R_1)v^n$	R_2v^{n+1}	.....	R_nv^{2n-1}	$(C+R_1)v^{2n}$	...

Assuming that the item has no resale price at the time of replacement, the present worth of all future discounted costs associated with the policy of replacing the item at the end of every n years will be given by

$$P(n) = [(C+R_1) + R_2v + \ldots + R_nv^{n-1}] + [(C+R_1)v^n + R_2v^{n+1} + \ldots + R_nv^{2n-1}]$$
$$+ [(C+R_1)v^{2n} + R_2v^{2n+1} + \ldots + R_nv^{3n-1}] + \ldots \text{and so on.}$$

Summing up the right-hand side, we get

$$P(n) = (C+R_1)(1 + v^n + v^{2n} + \ldots) + R_2v(1 + v^n + v^{2n} + \ldots) + \ldots + R_nv^{n-1}(1 + v^n + v^{2n} + \ldots)$$
$$= (C + R_1 + R_2v + \ldots\ldots + R_nv^{n-1})(1 + v^n + v^{2n} + \ldots)$$
$$= (C + R_1 + R_2v + \ldots + R_nv^{n-1}).\frac{1}{1-v^n} \quad [\because v < 1, \text{the sum of infinite G.P. is } 1/(1-v^n)] \quad \ldots(22.8a)$$

$$\therefore \qquad P(n) = \frac{F(n)}{1-v^n},\ P(n+1) = \frac{F(n+1)}{1-v^{n+1}} \qquad \ldots(22.8b)$$

where, for simplicity, $F(n) = C + R_1 + \ldots + R_nv^{n-1}$.

Step 2. ***To determine replacement policy so that P(n) is minimum.***

Since n is measured in discrete units, we shall use the method of finite differences in order to minimize the present worth expenditure $P(n)$.

Obviously, if $P(n+1) > P(n) > P(n-1)$, *i.e.* $\Delta P(n) > 0 > \Delta P(n-1)$, then $P(n)$ will be minimum.

So by definition of first difference

$$\Delta P(n) = P(n+1) - P(n) = \frac{F(n+1)}{1-v^{n+1}} - \frac{F(n)}{1-v^n} \quad \text{[from the equation (22.8b)]}$$

$$= \frac{F(n+1)(1-v^n) - F(n)(1-v^{n+1})}{(1-v^{n+1})(1-v^n)} \quad \left(\frac{N^r}{D^r}\text{ form}\right)$$

For convenience, we first simplify the N^r of $\Delta P(n)$ only. That is,

$$N^r = F(n+1)(1-v^n) - F(n)(1-v^{n+1})$$
$$= [F(n+1) - F(n)] + v^{n+1}F(n) - v^n F(n+1)$$
$$= R_{n+1}v^n + v^{n+1}F(n) - v^n[F(n) + v^n R_{n+1}] \quad [\because F(n+1) = F(n) + R_{n+1}v^n]$$
$$= v^n(1-v^n)R_{n+1} - v^n(1-v)F(n)$$

$$\therefore \quad \Delta P(n) = \frac{v^n(1-v^n)R_{n+1} - v^n(1-v)F(n)}{(1-v^{n+1})(1-v^n)} = \frac{v^n(1-v)}{(1-v^{n+1})(1-v^n)}\left[\frac{1-v^n}{1-v}R_{n+1} - F(n)\right] \quad ...(22.9)$$

Simply setting $n-1$ for n in (22.9),

$$\Delta P(n-1) = \frac{v^{n-1}(1-v)}{(1-v^n)(1-v^{n-1})}\left[\frac{1-v^{n-1}}{1-v}R_n - F(n-1)\right]$$

After little simplification of R.H.S. (*see foot-note*)*

$$\Delta P(n-1) = \frac{v^{n-1}(1-v)}{(1-v^n)(1-v^{n-1})}\left[\frac{1-v^n}{1-v}R_n - F(n)\right] \quad ...(22.10)$$

The quantity $v^n(1-v)/(1-v^{n+1})(1-v^n)$ in equation (22.9) is always positive, since $|v| < 1$. Thus, $\Delta P(n)$ has the same sign as the quantity under bracket [...] in (22.9), with similar explanation for $\Delta P(n-1)$ in (22.10) also.

Hence the condition, $\Delta P(n-1) < 0 < \Delta P(n)$, for minimum present worth expenditure becomes

$$\frac{1-v^n}{1-v}R_n - F(n) < 0 < \frac{1-v^n}{1-v}R_{n+1} - F(n) \quad ...(22.11a)$$

or
$$\frac{1-v^n}{1-v}R_n < F(n) < \frac{1-v^n}{1-v}R_{n+1} \quad ...(22.11b)$$

or
$$R_n < \frac{C + R_1 + R_2 v + \ldots + R_n v^{n-1}}{1 + v + v^2 + \ldots + v^{n-1}} < R_{n+1} \quad ...(22.12a)$$

or
$$R_n < \frac{F(n)}{\Sigma v^{n-1}} < R_{n+1}. \quad ...(22.12b)$$

The expression between R_n and R_{n+1} in (22.12) above is called the ***'weighted average cost'*** of previous n years with weights $1, v, v^2, \ldots, v^{n-1}$, respectively.

The value of n satisfying the relationship (22.11) or (22.12) will be the best replacement age of the item. This proves the theorem.

Q. 1. Describe the problem of replacement of items whose maintenance costs increase with time. You may assume that the money value also changes with time. **[Agra 93]**

2. Derive the following rules for minimizing costs in case of replecement of item whose maintenance costs increase with time :

(*i*) Replace if the next period's cost is greater than the weighted average of previous costs.

(*ii*) Do not replace if the next period's cost is less than the weighted average of previous costs. **[Raj. (M. Phil) 92]**

3. Find the optimum replacement policy which minimizes the total of all future discounted costs for an equipment which costs Rs. a and which needs maintenance costs of Rs. $C_1, C_2, \ldots C_n$ etc. ($C_{n+1} > C_n$) during the first year, second year, etc. respectively, and further d is the depreciation value per unit of money during a year.

4. What is discount rate ? If C is the capital cost of machine, R_i is the running cost in the ith year, v is the discount rate, how will you determine the best period r to replace the machine ? **[Garhwal M.Sc. (Stat.) 92]**

$$^*\ \frac{1-v^{n-1}}{1-v}R_n - F(n-1) = \frac{1-v^{n-1}}{1-v}R_n - [F(n) - R_n v^{n-1}] = \left(\frac{1-v^{n-1}}{1-v} + v^{n-1}\right)R_n - F(n) = \frac{1-v^n}{1-v}R_n - F(n).$$

Special Case :

It is interesting to note here that the replacement policy given by eqn. (22.7) (when money value was not counted) is a limiting case of that given by eqn. (22.12) (when money value is considered). Because, if the interest rate $r \to 0$, then $v \to 1$. Thus taking limit of (22.12) as $v \to 1$, we get

$$R_{n+1} > \frac{C + R_1 + R_2 + \dots + R_n}{1 + 1 + 1 + \dots + 1 \ (n \text{ times})} > R_n$$

i.e., $R_{n+1} > P(n)/n > R_n$ (when scrap value S is zero) which is identical to the result (22.7).

Second Method. Under the assumptions stated in *first Method*, we suppose that the amount equal to the present worth expenditure $P(n)$ is taken as loan (borrowed) at the rate of $r\%$ per year and this amount $P(n)$ is repaid by fixed annual payments throughout the life of the machine. Thus the variable payments actually made are converted into fixed annual payments x. Therefore, the present worth of fixed annual payments (x) for each cycle of n years becomes equal to the sum borrowed. That is, if $v = 1/(1+r)$, we have

$$P(n) = [x + vx + \dots + v^{n-1}x] + [v^n x + v^{n+1}x + \dots + v^{2n-1}x] + [v^{2n}x + v^{2n+1}x + \dots + v^{3n-1}x] + \dots,$$

$$= x(1 + v + \dots + v^{n-1}) + v^n x\,(1 + v + \dots + v^{n-1}) + v^{2n}x(1 + v + \dots + v^{n-1}) + \dots$$

$$= (1 + v + \dots + v^{n-1})\,(x + v^{2n}x + \dots) = \frac{1 - v^n}{1 - v} \times \frac{x}{1 - v^n}$$

$$\therefore \quad P(n) = \frac{x}{1 - v} \quad \text{or} \quad x = (1 - v)P(n) \qquad \text{...(22.13)}$$

Since $(1 - v)$ is positive constant, x will be minimum when $P(n)$ is minimum. The condition for $P(n)$ being minimum has already been obtained in *Step 2* of *first method*.

It is interesting to observe from eqn. (22.13) and (22.8b) that

$$x = (1 - v)\,P(n) = (1 - v).\frac{F(n)}{1 - v^n} = \frac{F(n)}{(1 - v^n)/(1 - v)} \qquad \text{...(22.14a)}$$

or

$$x = \frac{C + R_1 + R_2 v + \dots + R_n v^{n-1}}{1 + v + v^2 + \dots + v^{n-1}} = \frac{F(n)}{\Sigma v^{n-1}} \quad \left(\because 1 + v + v^2 + \dots + v^{n-1} = \frac{1 - v^n}{1 - v} \right) \qquad \text{...(22.14b)}$$

$\therefore$ Annual payment (x) = weighted average cost for n years. **[Meerut 98]**

Now proceed to determine a policy for selecting an economically best item from amongst those available from various factories.

22.9. HOW TO SELECT THE BEST MACHINE ?

In the problem of choosing a best machine (or item), the costs that are constant over time for each given machine will still have to be taken into account, although these costs may differ for each machine. Only those costs that are same for the machines under comparison can be excluded.

Suppose two machines M_1 and M_2 are at our choice. The data required for determining the best replacement age of each type of machine is also given from past experience. Thus, a best selection can be done by adopting the following outlined procedure :

Step 1. First find the best replacement age for machine M_1 and M_2 both by using the relationship :

$$R_{n+1} > \frac{F(n)}{\Sigma v^{n-1}} > R_n$$

Suppose the optimum replacement age for machines M_1 and M_2 comes out be n_1 and n_2, respectively.

Step 2. Compute the fixed annual payment (or weighted average cost) for each machine by using the formula :

$$x = \frac{C + R_1 + R_2 + \dots + R_n v^{n-1}}{1 + v + v^2 + \dots + v^{n-1}} = \frac{F(n)}{\Sigma v^{n-1}}$$

and substituting in this formula $n = n_1$ for machine M_1 and $n = n_2$ for machine M_2. Let it be x_1 and x_2 for machines M_1 and M_2, respectively.

Step 3. (*i*) If $x_1 < x_2$, then choose machine M_1. (*ii*) If $x_1 > x_2$, then choose machine M_2.

(*iii*) If $x_1 = x_2$, then both machines are equally good.

Note : This method can be extended to any number of machines, say $M_1, M_2, ..., M_k$. In general, first find the best replacement ages and corresponding weighted average costs $x_1, x_2, ..., x_k$ for all k machines, respectively. Then select the machine corresponding to least weighted average cost. The following illustrated practical example will make the procedure clear.

22.10 ILLUSTRATIVE EXAMPLE

Example 11. *A manufacturer is offered two machines A and B. A is priced at Rs. 5000 and running costs are estimated at Rs. 800 for each of the first five years, increasing by Rs. 200 per year in the sixth and subsequent years. Machine B, which has the same capacity as A, costs Rs.2500 but will have running costs of Rs. 1200 per year for six years, increasing by Rs. 200 per year thereafter.*

If money is worth 10% per year, which machine should be purchased ? (Assume that machines will eventually be sold for scrap at a negligible price. **[JNTU (B. Tech.) 2003]**

Solution. Since money is worth 10% per year, discount rate is given by $v = 1/(1 + 0.10) = 0.9091$.

Therefore, the optimum replacement age n must satisfy the relationship

$$R_{n+1} > \frac{C + R_1 + R_2 v + \ldots + R_n v^{n-1}}{1 + v + v^2 + \ldots + v^{n-1}} > R_n \quad \text{or} \quad R_{n+1} > \frac{F(n)}{\Sigma v^{n-1}} > R_n,$$

for the first time.

For the best replacement age n, tabulate the required calculations of machine A (see *Table* 22.9)

Table 22.9

Year (n)	Running cost (R_n)	Pwf (v^{n-1})	$v^{n-1}R_n$	F(n) $= C + \Sigma R_n v^{n-1}$	Σv^{n-1}	Weighted Average $F(n)/\Sigma v^{n-1}$
1	800	1.0000	800	5,800	1.0000	5800
2	800	0.9091	727	6,527	1.9091	3419
3	800	0.8264	661	7,188	2.7355	2628
4	800	0.7513	601	7,789	3.4868	2234
5	800	0.6830	546	8,335	4.1698	1999
6	1000	0.6209	621	8,956	4.7907	1896
7	1200	0.5645	677	9,633	5.3552	1799
8	1400	0.5132	718	10,351	5.8684	1764
→9	1600	0.4665	746	11,097	6.3349	1752
10	1800	0.4241	763	11,860	6.7590	1755

It is observed that $R_{10} = 1800$ become greater than the weighted average cost Rs. 1752 for nine years.

Hence, it would be best to replace machine A after nine years.

Also, from *Table 22.9*, the equivalent fixed annual payment for machine A is read from last column as

$x_1 =$ weighted average for 9 years = Rs. 1752. (***Note***)

In a similar fashion, best replacement age for machine B can be calculated. Calculations for machine B are given in Table 22.10.

Since $R_9 >$ weighted average for 8 years (*i.e.* 1800 > 1680), it would be best to replace machine B after 8 years.

The equivalent fixed annual payment for machine B is read from *Table* 22.10 as

$$x_2 = \text{weighted average for 8 years} = \text{Rs. } 1680.$$

Table 22.10.

Year (n)	Running cost (R_n)	Pwf (v^{n-1})	$v^{n-1}R_n$	$F(n) = C + \Sigma R_n v^{n-1}$	Σv^{n-1}	Weighted Average $F(n)/\Sigma v^{n-1}$
1	1200	1.0000	1200	3700	1.0000	3700
2	1200	0.9091	1091	4791	1.9091	2510
3	1200	0.8264	991	5782	2.7355	2114
4	1200	0.7513	902	6684	3.4868	1917
5	1200	0.6830	820	7504	4.1698	1800
6	1200	0.6209	745	8249	4.7907	1722
7	1400	0.5645	790	9039	5.3552	1688
→8	1600	0.5132	821	9860	5.8684	1680
9	1800	0.4665	840	10700	6.3349	1689

Since x_1 is greater than x_2, it would be better to purchase machine B instead of A, although *the average of actual payments* (without considering money value) is Rs. 1578 for A and Rs. 1588 for B.

EXAMINATION PROBLEMS

1. If you wish to have a return of 10% per annum one your investment, which of the following plan would you prefer ?

	Plan A	Plan B
First cost (Rs.) :	2,00,000	2,50,000
Scrap value for 15 years :	1,50,000	1,80,000
Excess of annual revenue over annual disbursement :	25,000	30,000

[**Hint.** Proceed exactly as solved Example 6.]

2. Assume that present value of one rupee to be spent in a year's time is Re. 0.90 and C = Rs. 3,000 capital of equipment and the running costs are given in the table below :

Year :	1	2	3	4	5	6	7
Running cost (Rs.) :	500	600	800	1000	1300	1600	2000

When should the machine be replaced ?
[**Hint.** Here $v = 0.9$, C = Rs. 3,000, R_n (n = 1, 2, ..., 7) given above. Compute the table and find 1300 < 1569 < 1600.]
[**Ans.** It is always better to replace the machine after 5 years.]

3. The cost of a new machine is Rs. 5000. The maintenance cost of nth year is given by $R_n = 500\ (n-1), n = 1, 2, \ldots$ .Suppose that the discount rate per year is 0.5. After how many years it will be economical to replace the machine by a new one ? Derive the formula you use in your problem. **[Meerut 96, 93P]**
[**Hint.** Here $v = 0.5$, C = Rs.5000, $R_1 = 0$, $R_2 = 500$, $R_3 = 100$, and so on. Proceed as in Table 4.9. Observe 2500 < 2992 < 2992.13.]
[**Ans.** replace after 6 years.]

4. A truck is priced at Rs. 60,000 and running costs are estimated at Rs. 6000 for each of the first four years, increasing by Rs. 2000 per year in the fifth and subsequent years. If money is worth 10 per cent per year, when should the truck be replaced ?
Assume that truck will eventually be sold for scrap at negligible price.
[**Ans.** Replace after 9 years.]

5. If I wish to have a minimum rate of return of 10% per annum on my investment, which of the following two plans should I prefer ?

	Plan A	**Plan B**
First cost :	75,000	75,000
Estimated scrap value after 20 yrs. :	37,500	6,000
Excess of annual receipts over annual disbursements :	7,500	9,000

Annual series Pwf's at 10% for 20 years = 8.514. Single payment Pwf's at 10% for 20 years = 0.2472.

6. A truck has been purchased at a cost of Rs. 1,60,000. The value of the truck is depreciated in the first three years by Rs. 20,000 each year and Rs. 16,000 per year thereafter. Its maintenance and operating costs for the first three years are Rs. 16,000, Rs. 18,000 and Rs. 20,000 in that order and increase by Rs. 4,000 every year.
Assuming an interest rate of 10% find the economic life of the truck.
[**Ans.** 7 years.]

7. An engineering company is offered two types of material handling equipment A and B. A is priced at Rs. 60,000 including cost of installation and the cost for operation and maintenance are estimated to Rs. 10,000 for each of the first five years, increasing by Rs. 3,000 per year in the sixth and subsequent years. Equipment B with a related capacity same as A, requires an initial investment of Rs. 30,000 but in terms of operation and maintenance cost more than A. These costs for B are estimated to be 13,000 per year for the first six years, increasing by Rs. 4,000 per year for each year from the 7th year onwards. The company expects a return of 10 per cent on all its investments. Neglecting the scrap value of the equipment at the end of its economic life, determine which equipment company should buy.
[**Ans.** Purchase B. Replace it after 7 years.]

8. A fleet owner finds from his past records that the costs per year of running a truck whose purchase price is Rs. 8,000 are as given below :

Year :	1	2	3	4	5	6	7	8
Running cost :	1000	1500	1600	1700	2000	3000	3200	4300
Resale price :	2000	1500	1000	800	500	400	300	300

A new type of truck is announced in the market, with 50% more capacity than the old one at a unit price of Rs. 7,500. He estimates the running costs and resale price for the new truck as follows :

Year :	1	2	3	4	5	6	7	8
Running cost :	1000	1500	1700	2300	3000	3900	5000	6000
Resale Price :	3700	3000	2100	1000	700	500	300	300

Assume that loss of flexibility due to fewer trucks is of no importance and that he will continue to have sufficient work for the old trucks, should he go in for a new one or not ? [Objective is to minimize the discounted value of all future costs : Discount factor = $(1.1)^{-n}$]

$n =$	1	2	3	4	5	6	7	8	9
$(1.1)^{-n} =$	0.9091	0.8264	0.7513	0.683	0.6209	0.5654	0.5132	0.4665	0.4241

[Ans. It is better to replace old truck by new one.**]**

II-Replacement of Items that Fail Completely

22.11. INDIVIDUAL REPLACEMENT POLICY : *MORTALITY THEOREM*

Under this policy an item is immediately replaced as soon as it fails. To determine the probability distribution of failure (or life span of any item) its mortality tables are used.

To discuss such type of replacement policy, we consider the problem of human population. No group of people ever existed under the conditions :

(*i*) that all deaths are immediately replaced by births, and (*ii*) that there are no other entries or exits.

But, the reason for stating the problem under these two assumptions is that the analysis becomes more easy by keeping the virtual human population in mind. Later, such problems can also be translated into industrial items, where death is equivalent to a part failure and birth is equivalent to new replacement. Thus, industries also face a fairly common situation. The following ***Mortality Theorem*** will make the conceptions clear.

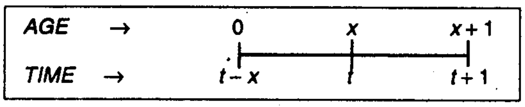
Theorem 22.3. (Mortality Theorem). *A large population is subject to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no other entries or exits. Show that the age distribution ultimately becomes stable and that the number of deaths per unit time becomes constant (which is equal to the size of the total population divided by the mean age at death).*

[Meerut 2002, 98; Garhwal M.Sc. (Stat.) 93]

Proof. For convenience, we suppose that each death occurs just before some time $t = w$, where w is an integer and that no member of the population remains alive longer than $w + 1$ time units.

Let $f(t)$ = number of births at time t, $p(x)$ = probability of member dying just before age $x + 1$, *i.e.* at the age x.

Since $f(t - x)$ represents the number of births at time $t - x$, the age of those members who remain alive at time t will obviously be x. This can be easily understood by the following diagram.



So their former probability of dying at time t (just before time $t + 1$) will be equal to the probability $p(x)$ of those dying at age x (just before age $x + 1$).

Hence, the expected number of deaths of such alive members at time t is $p(x)\, f(t - x)$.

Therefore, total number of deaths at time t will be

$$= \sum_{x=0}^{w} f(t-x)\, p(x), \quad t = w, w+1, w+2, \ldots \ldots$$

Also, total number of births at time $(t + 1) = f(t + 1)$.

Since all deaths at time t are immediately replaced by births at time $t + 1$, therefore

$$f(t+1) = \sum_{x=0}^{w} f(t-x)\, p(x), \qquad \ldots(22.15)$$

The difference eqn. (22.15) in t may be solved by substituting $f(t) = A\alpha^t$.

Then the difference equation (22.15) becomes

$$A\,\alpha^{t+1} = A \sum_{x=0}^{w} \alpha^{t-x} p(x)$$

On dividing by $A\,\alpha^{t-w}$, we get $\alpha^{w+1} = \sum_{x=0}^{w} \alpha^{w-x} p(x)$ or $\alpha^{w+1} - \sum_{x=0}^{w} \alpha^{w-x} p(x) = 0$

or $$\alpha^{w+1} - [\alpha^{w} p(0) + \alpha^{w-1} p(1) + \alpha^{w-2} p(2) + \ldots + p(w)] = 0. \qquad \ldots(22.16)$$

Since the sum of all probabilities is unity, so

$$\sum_{x=0}^{w} p(x) = 1 \text{ or } 1 - \sum_{x=0}^{w} p(x) = 0 \text{ or } 1 - [p(0) + p(1) + p(2) + \ldots + p(w)] = 0. \quad ...(22.17)$$

Now comparing equations (22.16) and (22.17), it is found that one solution of (22.16) is $\alpha_0 = 1$. But the polynomial equation (22.16) must have $w + 1$ total number of roots. Let the remaining roots be denoted by $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_w$. Consequently, the solution of difference eqn. (22.15) will be of the form :

$$f(t) = A_0 + A_1\alpha_1^t + A_2\alpha_2^t + \ldots + A_w\alpha_w^t \quad ...(22.18)$$

where $A_0, A_1, A_2, \ldots, A_w$ are constants whose value can be determined with the help of age distribution at some given point in time. Further, it can be shown (**see foot note**)* that the absolute value of all the remaining roots is less than unity, *i.e.*

$$|\alpha_i| < 1, \text{ for } i = 1, 2, 3, \ldots, w.$$

Hence, $\alpha_1^t, \alpha_2^t, \alpha_3^t, \ldots, \alpha_w^t$ tends to zero as $t \to \infty$. Consequently, equation (22.18) becomes $f(t) = A_0$, which shows that the number of deaths per unit time (as well as the number of births) is constant at A_0.

To show that the age distribution ultimately becomes stable :

Let the probability of members remain alive longer than x time units be $P(x)$, then

$$P(x) = 1 - \{p(0) + p(1) + p(2) + \ldots + p(x-1)\} \text{ and } P(0) = 1. \quad ...(22.19)$$

Since births and deaths have settled down to constant rate A_0, the expected number of survivors of age x is also stable at $A_0 P(x)$.

Since the number of births are always equal to the number of deaths, the size N of total population is constant, *i.e.*,

$$N = A_0 \sum_{x=0}^{w} P(x) \quad ...(22.20) \qquad \text{or} \qquad A_0 = \frac{N}{\sum_{x=0}^{w} P(x)} \quad ...(22.21)$$

Now the number of survivors aged 0, 1, 2, 3, ... can be computed from equation (22.20) as A_0, $A_0P(1)$, $A_0P(2)$, and so on.

Finally, if the denominator in (22.21), *i.e.* $\sum_{x=0}^{w} P(x)$, is equivalent to mean age at death, then the age distribution will ultimately become stable.

To prove this, $\sum_{x=0}^{w} P(x) = \sum_{x=0}^{w} P(x)\, \Delta(x)$ [since $\Delta(x) = (x+1) - x = 1$ by finite differences]

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* Restating the equaitons (22.16) and (22.17) we have

$$\alpha^{w+1} - p(0)\alpha^{w} - p(1)\alpha^{w-1} - p(2)\alpha^{w-2} - \ldots - p(w-1)\alpha - p(w) = 0 \quad ...(1)$$

and

$$1 - p(0) - p(1) - p(2) - \ldots - p(w-1) - p(w) = 0 \quad ...(2)$$

Let $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{w-1}, \alpha_w$ be the roots of (1) where $\alpha_0 = 1$ by virtue of (2).

By theory of equaions, sum of all the roots of (1) is given by

$$\alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_{w-1} + \alpha_w = \frac{p(0)}{1}$$

or

$$1 + \alpha_1 + \alpha_2 + \ldots + \alpha_{w-1} + \alpha_w = p(0) \quad (\because \alpha_0 = 1)$$

or

$$\alpha_1 + \alpha_2 + \ldots + \alpha_{w-1} + \alpha_w = p(0) - 1. \quad ...(3)$$

But, the *Descarte's Rule of Signs* states '**The number of positive roots of the polynomial equaion $f(x) = 0$ cannot exceed the number of changes of sign (from + ive to– ive or from –ive to + ive) in the terms occurring in $f(x)$**'.

Applying this rule we find that there is only one change of sign from + ive to – ive in the equation (1), so there can be not more than one +ive root. As already determined, one positive root is $\alpha_0 = 1$.Hence all remaining roots $\alpha_1, \alpha_2, \ldots, \alpha_w$ will be negaive. So considering the absolute value on both sides of (3), we have

$$|\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{w-1} + \alpha_w| = |p(0) - 1| < 1 \qquad [\because 0 < p(0) < 1]$$

or

$$|\alpha_1| + |\alpha_2| + |\alpha_3| + \ldots + |\alpha_{w-1}| + |\alpha_w| < 1.$$

Since the sum of positive numbers is less than 1, each number will also be individually less than 1.

$$= [P(x).x]_0^{w+1} - \sum_{x=0}^{w} (x+1)\,\Delta P(x) \qquad \text{(using formula (2.17) on page 39)}$$

$$= [P(w+1).(w+1) - 0] - \sum_{x=0}^{w} (x+1)\,\Delta P(x) \qquad ...(22.22)$$

But, $P(w+1) = 1 - p(0) - p(1) - p(2) - \ldots - p(w)$ [from eqn. (22.19)]

$= 0$ [by virtue of eqn. (22.17)]

and $\Delta P(x) = P(x+1) - P(x)$

$= [1 - p(0) - p(1) - p(2) - \ldots - p(x)] - [1 - p(0) - p(1) - p(2) - \ldots - p(x-1)]$

$= -p(x).$

Therefore, substituting the simplified values of $P(w+1)$ and $\Delta P(x)$ in eqn. (4.22) to obtain

$$\sum_{x=0}^{w} P(x) = 0 + \sum_{x=0}^{w} (x+1)\,p(x) = \sum_{y=1}^{w+1} y\,p(y-1) \qquad [\text{setting } x+1 = y]$$

$$= \sum_{y=1}^{w+1} y \times \text{prob.[that age at death is } y] = \textbf{mean (expected) age at death.}$$

This completes the proof of the theorem.

22.12. GROUP REPLACEMENT OF ITEMS THAT FAIL COMPLETELY

Group replacement is concerned with those items that either work or fail completely. It often happens that a system contains a large number of identical low cost items that are increasingly liable to failure with age. In such cases, there is a set-up cost for replacement that is independent of the number replaced and it may be advantageous to replace all items at fixed intervals. Such a policy is called ***group replacement*** and is particularly attractive when the value of any individual item is so small that the cost of keeping records of individual ages cannot be justified. The classical example of such a policy is that used in replacing light bulbs.

Two types of replacement policies are considered :

1. ***Individual replacement.*** Under this policy, an item is replaced immediately after its failure.
2. **Group replacement.** Under this policy, decision is taken as to when all the items must be replaced, irrespective of the fact that items have failed and have not failed, with the provision that if any item fails before the optimal time, it may be replaced individually.

22.12-1. Group Replacement Policy

Group replacement policy is defined in the following theorem and later it is explained by numerical examples.

Theorem 22.4. (Group replacement policy) :

(*a*) *One should group replace at the end of t^{th} period if the cost of individual replacements for the t^{th} period is greater than the average cost per period through the end of t^{th} period.*

(*b*) *One should not group replace at the end of t^{th} period if the cost of individual replacements at the end of $(t-1)^{th}$ period is less than the average cost per period through the end of t^{th} period.* **[Meerut 95]**

Proof. Here it is proposed to replace all items at fixed interval 't', whether they have failed or not, and continue replacing failed items as and when they fail.

Let N_t = number of units failing during time t

N = total number of units in the system

$C(t)$ = cost of group replacement after time period t

C_1 = individual replacement cost on failure

C_2 = per unit cost of replacement in a group.

Then $C(t) = C_1\,[N_1 + N_2 + \ldots + N_{t-1}] + C_2 N$. Therefore, average cost per unit period will be

$$F(t) = \frac{C(t)}{t} = \frac{C_1\,[N_1 + N_2 + \ldots + N_{t-1}] + C_2 N}{t} \qquad ...(22.23)$$

Now in order to determine the replacement age 't', the average cost per unit period [$C(t)/t = F(t)$, say] should be minimum.

The condition for minimum of $F(t)$ is

$$\Delta F(t-1) < 0 < \Delta F(t) \quad \text{...(22.24)}$$

Now, $$\Delta F(t) = F(t+1) - F(t) = \frac{C(t+1)}{t+1} - \frac{C(t)}{t} = \frac{C(t) + C_1 N_t}{t+1} - \frac{C(t)}{t}$$

$$= \frac{t C_1 N_t - C(t)}{t(t+1)} = \frac{C_1 N_t - C(t)/t}{t+1} \quad \text{...(22.25)}$$

which must be greater than zero for minimum $F(t)$, that is

$$C_1 N_t > C(t)/t \quad \text{...(22.26)}$$

Similarly, from $\Delta F(t-1) < 0$

$$C_1 N_{t-1} < C(t)/t. \quad \text{...(22.27)}$$

Thus, from equations (22.26) and (22.27), the group replacement policy is completely established.

Q. 1. Explain how the theory of replacement is used in the problem of replacement of items that fail completely. **[Raj. Univ. (M. Phil.) 90]**

2. Briefly explain what do you mean by individual and group replacement policy. **[JNTU (B. Tech.) 2003]**

3. Explain group replacement concept and its applications. **[JNTU (Mech. & Prod.) 2004]**

22.13. ILLUSTRATIVE EXAMPLES

Example 12. *Following failure rates have been observed for a certain type of light bulbs :*

Week	1	2	3	4	5
Per cent failing by the end of week :	10	25	50	80	100

There are 1000 bulbs in use, and it costs Rs. 10 to replace an individual bulb which has burn out. If all bulbs were replaced simultaneously it would cost Rs. 4 per bulb. It is proposed to replace all bulbs at fixed intervals of time, whether or not they have burnt out, and to continue replacing burnt out bulbs as and when they fail. At what intervals all the bulbs should be replaced ? At what group replacement price per bulb would a policy of strictly individual replacement become preferable to the adopted policy ?

[JNTU (B. Tech.) 2003, 02; Meerut 2002; AIMA, (P.G. Dip. in Management) June 97; Garhwal M.Sc. (Stat.) 95]

Solution. Let p_i be the probability that a light bulb, which was new when placed in position for use, fails during the ith week of its life.

Thus, following frequency distribution is obtained assuming to replace burnt out bulbs as and when they fail.

$p_1 =$ the prob. of failure in Ist week $= 10/100 = 0.10$

$p_2 =$ the prob. of failure in IInd week $= (25-10)/100 = 0.15$

$p_3 =$ the prob. of failure in IIIrd week $= (50-25)/100 = 0.25$

$p_4 =$ the prob. of failure in IVth week $= (80-50)/100 = 0.30$

$p_5 =$ the prob. of failure in Vth week $= (100-80)/100 = 0.20$

Sum of all prob. $= 1.00$

Since the sum of all probabilities can never be greater than unity, therefore all further probabilities p_6, p_7, p_8, and so on, will be zero. Thus, a bulb that has already lasted four weeks is sure to fail during the fifth week.

Furthermore, assume that

(*i*) bulbs that fail during a week are replaced just before the end of that week, and

(*ii*) the actual percentage of failures during a week for a subpopulation of bulbs with the same age is the same as the expected percentage of failures during the week for that subpopulation.

Let N_i be the number of replacements made at the end of the ith week, if all 1000 bulbs are new initially.

Thus,

$N_0 = N_0 \quad = 1000$

$N_1 = N_0 p_1 = 1000 \times 0.10 \quad = 100$

$N_2 = N_0 p_2 + N_1 p_1 = 1000 \times 0.15 + 100 \times 0.10 \quad = 160$

$$N_3 = N_0p_3 + N_1p_2 + N_2p_1 = 1000 \times 0.25 + 100 \times 0.15 + 160 \times 0.10 = 281$$
$$N_4 = N_0p_4 + N_1p_3 + N_2p_2 + N_3p_1 = 377$$
$$N_5 = N_0p_5 + N_1p_4 + N_2p_3 + N_3p_2 + N_4p_1 = 350$$
$$N_6 = 0 + N_1p_5 + N_2p_4 + N_3p_3 + N_4p_2 + N_5p_1 = 230$$
$$N_7 = 0 + 0 + N_2p_5 + N_3p_4 + N_4p_3 + N_5p_2 + N_6p_1 = 286$$

and so on.

It has been observed that expected number of bulbs burnt out in each week increases until 4th week and then decreases until 6th week and again starts increasing. Thus, the number will continue to oscillate and ultimately the system settles down to a steady state in which the proportion of bulbs failing in each week is the reciprocal of their average life.

Since the mean age of bulbs

$$= 1 \times p_1 + 2 \times p_2 + 3 \times p_3 + 4 \times p_4 + 5 \times p_5$$
$$= 1 \times 0.10 + 2 \times 0.15 + 3 \times 0.25 + 4 \times 0.30 + 5 \times 0.20 = 3.35 \text{ weeks},$$

the number of failures in each week in steady state becomes

$$= 1000/3.35 = 299$$ [by ***Mortality Theorem**, Eqn.* (22.21)]

and the cost of replacing bulbs individually only on failure

$$= 10 \times 299 \text{ (at the rate of Rs. 10 per bulb)} = \text{Rs. } 2990.$$

Since the replacement of all 1000 bulbs simultaneously costs Rs. 4 per bulb and replacement of an individual bulb on failure costs Rs. 10, therefore cost of replacement of all bulbs simultaneously is as given in the following table :

End of week	Cost of individual replacement	Total cost of group replacement (Rs.)	Average cost per week (Rs.)
1	$100 \times 10 = 1000$	$1000 \times 4 + 100 \times 10 = 5000$	5000.00
2	$160 \times 10 = 1600$	$5000 + 160 \times 10 = 6600$	3300.00
3	$281 \times 10 = 2810$	$6600 + 281 \times 10 = 9410$	**3136.67**
4	$377 \times 10 = $ **3770**	$9410 + 377 \times 10 = 13180$	3295.00

The cost of individual replacement in the *fourth* week exceeds the average cost for three weeks.

Thus it would be optimal to replace all the bulbs after every three weeks, otherwise the average cost will start increasing.

Further, since the group replacement at the end of one week costs Rs. 5000 and the individual replacement after one week costs Rs. 2990, the individual replacement will be preferable.

Example 13. *A computer contains 10,000 resistors. When any one of the resistor fails, it is replaced. The cost of replacing a single resistor is Rs. 10 only. If all the resistors are replaced at the same time, the cost per resistor would be reduced to Rs. 3.50. The per cent surviving by the end of month t is as follows :*

Month (t) :	0	1	2	3	4	5	6
% surviving by the end of month :	100	97	90	70	30	15	0

What is the optimum plan ? **[Meerut 2005, 99; JNTU (B. Tech.) 2003; Agra 93]**

Solution. The probabilities p_i of failure during the month t are

$p_1 = (100 - 97)/100 = 0.03$ $\quad p_4 = (70 - 30)/100 = 0.40$

$p_2 = (97 - 90)/100 = 0.07$ $\quad p_5 = (30 - 15)/100 = 0.15$ $\quad \sum_{i=1}^{6} p_i = 1.00$

$p_3 = (90 - 70)/100 = 0.20$ $\quad p_6 = (15 - 0)/100 = 0.15$

Now as in ***Example 12,*** we obtain

$N_0 = 10{,}000, N_1 = 300, N_2 = 709, N_3 = 2042, N_4 = 4{,}171, N_5 = 2{,}030, N_6 = 2{,}590.$

Average life of each resistor $= \sum_{i=1}^{6} i p_i = 4.02$ months.

Average number of replacements every month = 2488 (approx.)

Average cost of monthy individual replacement = Rs. 24880.

Under group replacement, we find :

End of month	Cost of individual replacement	Total cost of group replacement (Rs.)	Average cost per month
1	3000	10,000 × 3·50 + 300 × 10 = Rs.38000	38,000.00
2	7090	38000 + 709 × 10 = Rs.45090	22,545.00
→ 3	20420	45090 + 2,042 × 10 = Rs. 65510	21,836.66
4	41710	65510 + 4,171 × 10 = Rs. 107220	26,805.00

The cost of individual replacement in the fourth month exceeds the average cost for 3 months.

Hence, it would be optimal to replace all the resistors after every three months, otherwise the average cost will start increasing.

Example 14. *Truck tyres which fail in service can cause expensive accidents. It is estimated that a failure in service results in an average cost Rs. 1,000 exclusive of the cost of replacing the burst tyre. New tyres cost Rs. 400 each and are subject to mortality as in Table below. If the tyres are to be replaced after a certain fixed mileage or on failure (whichever occurs first), determine the replacement policy that minimizes the average cost per mile. Mention the assumptions you make to arrive at the solution.*

Table Showing Truck Tyre Mortality

Age of tyre at failure (Miles)	Proportion of tyre
≤ 10,000	0.000
10,001–12,000	0.020
12,001–14,000	0.035
14,001–16,000	0.063
16,001–18,000	0.100
18,001–20,000	0.220
20,001–22,000	0.345
22,001–24,000	0.205
24,001–26,000	0.012
	Total 1.000

Solution. ***Assumptions*** : (*i*) Failures take place at the exact ages 11,000, 13,000, 15,000 etc.

(*ii*) Initially, there are 1000 tyres.

(*iii*) Up to the age of 10,000 miles, proportion of tyre fail is zero. And from the age of 11,000 to 13,000 the proportion of tyres is 0.030. Assume that in this period the average cost of maintaining is 10,000. If in this period a tyre bursts, the cost will be Rs. 1,400. Thus the cost of individual replacement is Rs. 1,400. The cost of group replacement is given Rs. 400 per tyre in this problem.

$N_0 = 1000$, $N_1 = 20$, $N_2 = 35$, $N_3 = 64$, $N_4 = 103$, $N_5 = 28$, $N_6 = 357$, $N_7 = 229$, and so on.

Expected life $= \sum_{i=0}^{\infty} i p_i = 4.28$, Average number of failure per period $= 1{,}000/4.28 = 234$,

Cost of individual replacement per period $= 234 \times 1400$.

$\therefore$ Cost of individual replacement per mile $= \dfrac{234 \times 1400}{2000} = 164.$

For group replacement, compute the following :

End of Period (in miles)	Total cost of group replacement	Average cost per period in miles
(11,000 - 13,000) → 1	1000 × 400 + 20 × 1400 = 4,28,000	4,28,000
(13,000 - 15,000) → 2	2,28,000 + 35 × 1400 = 4,77,000	2,38,500
... ...	... ...	...
(19,000 - 21,000) → 5	7,10,800 + 28 × 1400 = 7,50,000	1,50,000
(21,000 - 23,000) → 6	7,20,000 + 357 × 1400 = 12,49,800	2,08,300

Group replacement is optimal after every 5th period, (i.e., after 1,70,000 mileage). So minimum cost per mile = Rs. 150.

Example 15. *A company manufactures automobile batteries at a factory cost of Rs. 10 each. Battery life is subject to mortality as in the following table. The company has a guarantee policy under which if a battery fails during the first month after purchase the refund of the full price or a new battery is made, if it fails during the second month a refund of 29/30 of the full price is given, during the third month 28/30 and so on until the 30th month after purchase, at which time the refund of 1/30 of full price is made. At what unit price batteries should be sold so that on an average the company will break even ? (Assume that battery age is zero at the time of purchase.)*

Table Showing the Mortality of Battery

Age in months	Probability of failure
0	0.050
1	0.000
2	0.000
3	0.000
4	0.000
5	0.000
6	0.000
7	0.000
8	0.000
9	0.000
10	0.000
11	0.000
12	0.000
13	0.000
14	0.000
15	0.000

Age in months	Probability of failure in next month
16	0.000
17	0.100
18	0.100
19	0.100
20	0.100
21	0.100
22	0.100
23	0.015
24	0.015
25	0.020
26	0.025
27	0.030
28	0.035
29	0.040
30	0.710

Solution. Let K be the break-even price, and p_i be the probability that a new battery will fail during the $i + 1$st month after purchase.

Average refund on batteries that fail will be

$$\left(1 \times p_1 + \frac{29}{30} \times p_2 + \frac{28}{30} \times p_3 + \ldots + \frac{1}{30} \times p_{29}\right) K = 0.0908 \quad \text{(substituting the values of } p_1, p_2, \ldots p_{29} \text{ from the table)}$$

But, K = factory cost + expected refund. Therefore, $K = 10 + 0.908K$

Ans. K = Rs. 11.00.

Example 16. *(a) At time zero, all items in a system are new. Each item has a probability $q (= 1 - p)$ of failing immediately before the end of the second month (all items fail by the end of the second month). If all the items are replaced as they fail, show that the expected number of failures $f(x)$ at the end of the month x will be*

$$f(x) = \frac{N}{(1 + q)} [1 - (-q)^{x+1}]$$

where N is the number of items in the system. **[Meerut 97P, 93; Garhwal (Stat.) 91]**

(b) If the cost per item of individual replacement is C_1 and the cost per item of group replacement is C_2, find the condition under which—

(i) a group replacement policy at the end of each month is the most profitable.

(ii) a group replacement policy at the end of every other month is the most profitable.

(*iii*) *no group replacement policy is better than a policy of pure individual replacement (i.e. in steady state condition).* **[Meerut 93; Garhwal M. Sc. (Stat.) 91]**

Solution. (a) Let N_i be the expected number of replacements (failures) made at the end of ith month, if N items are new initially. Then,

N_0 = number of items in the system initially $= N$

$N_1 = N_0 p = N(1-q)$

$N_2 = N_0 q + N_1 p = Nq + N_1(1-q) = Nq + N(1-q)^2 = N(1-q+q^2)$

$N_3 = N_1 q + N_2 p = N(1-q)q + N(1-q+q^2)(1-q) = N(1-q)(1+q^2) = N(1-q+q^2-q^3)$, and so on.

Suppose, $N_k = N[1-q+q^2-q^3+\ldots+(-1)^k q^k]$

Therefore, $N_{k+1} = N_{k-1}\,q + N_k\,p$

$$= N[1-q+q^2-\ldots+(-q)^{k-1}]\,q + N[1-q+q^2-\ldots+(-q)^k](1-q)$$
$$= N[1-q+q^2-q^3+\ldots+(-q)^{k+1}].$$

Hence, by induction, the expected number of replacements at the end of month x will be given by

$$f(x) = N[1-q+q^2-q^3+\ldots+(-q)^x]$$
$$= \frac{N[1-(-q)^{x+1}]}{1+q} \text{ [using formula for sum of GP]} \qquad \ldots(22.28)$$

This proves the first part of this example.

(*b*) Now, in steady state condition, the expected number of individual replacements

$$= \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{N}{1+q}[1-(-q)^{x+1}]$$
$$= N/(1+q) \qquad \text{(since } q<1,\ (-q)^{\infty}=0) \qquad \ldots(22.29)$$

Alternatively, by ***Mortality Theorem*** [eqn. (22.21)] the number of failures per unit time becomes

$$= \frac{N}{\text{Mean age at failure}} = \frac{N}{1.p+2.q} = \frac{N}{1+q} \qquad (\because\ p+q=1)$$

Therefore, average cost per month for pure individual replacement policy will be

$$= C_1N/(1+q). \qquad \ldots(22.30)$$

(*i*) The average cost per month for group replacement policy at the end of Ist month

$$= N_1C_1 + NC_2 = N(1-q)\,C_1 + NC_2 \qquad \ldots(22.31)$$

Thus, a group replacement policy at the end of first month will be better than pure individual replacement, if

$$N(1-q)\,C_1 + NC_2 < NC_1/(1+q) \qquad \text{[from (22.30) and (22.31)]}$$

or
$$C_2 < q^2C_1/(1+q). \qquad \ldots(22.32)$$

(*ii*) The average cost per month for group replacement policy at the end of 2nd month is

$= ½\,[N_1+N_2)\,C_1 + NC_2]$ (which is the average of first two months)

$= ½\,[N(1-q) + N(1-q+q^2)]\,C_1 + ½\,NC_2$

$= N[1-q+½\,q^2]\,C_1 + ½\,NC_2.$

Here group replacement policy will be better than pure individual replacement policy, if

$$N(1-q+½\,q^2)\,C_1 + ½\,NC_2 < NC_1/(1+q)$$

or
$$C_2 < q^2(1-q)\,C_1/(1+q) \qquad \ldots(22.33)$$

(*iii*) If no group replacement policy is better than a policy of pure individual replacement, then from equations (22.32) and (22.33),

$$C_2 > C_1q^2/(1+q), \text{ and } C_2 > C_1q^2(1-q)/(1+q) \qquad \ldots(22.34)$$

or
$$C_1 < (1+q)\,C_2/q^2, \text{ and } C_1 < (1+q)C_2/(1-q)q^2 \qquad \ldots(22.35)$$

Since $q<1,\ (1+q)/q^2 < (1+q)/q^2(1-q)$

Hence $C_1 < (1+q)C_2/q^2 \qquad \ldots(22.36)$

EXAMINATION PROBLEMS

1. The following failure rates have been observed for a certain type of light bulbs.

End of week	:	1	2	3	4	5	6	7	8
Prob. of failure to date	:	0.05	0.13	0.25	0.43	0.68	0.88	0.96	1.00

The cost of replacing an individual bulb is Rs. 2.25, the decision is made to replace all bulbs simultaneously at fixed intervals, and also to replace individual bulbs as they fail in service. If the cost of group replacement is 60 paise per bulb and the total number of bulbs is 1000, what is the best interval between group replacements ?

[VTU 2002; Agra 98; Rohil. 90; Raj. Univ. (M. Phil) 90]

[**Ans.** Replace at the end of 3rd week.]

2. The probability p_n of failure just before age n are shown below. If individual replacement costs Rs. 1.25 and group replacement costs Re. 0.50 per item, find the optimal group replacement policy.

$n = 1$	2	3	4	5	6	7	8	9	10	11
$p_n = .01$	.03	.05	.07	.10	.15	.20	.15	.11	.08	.05

[**Hint.** Assuming that there are 1000 items in use, proceed as in solved *Example 12.* $N_1 = 10$, $N_2 = 30$, $N_3 = 51$,. $N_4 = 72$, $N_5 = 105$, $N_6 = 168$.
Average costs are : 512.25, 274.87, 204.50, 175.87, 166.95, 172.37.]
[**Ans.** Replace after every 5 weeks.]

3. Suppose that a special purpose type of light bulb never lasts longer than 2 weeks. There is a chance of 0.3 that a bulb will fail at the end of first week. There are 100 new bulbs initially. The cost per bulb for individual replacement is Rs. 1.25 and the cost per bulb for a group replacement is Re. 0.50.

Is it cheapest to replace all bulbs : (*i*) individually, (*ii*) every week, (*iii*) every second week, (*iv*) every third week ?
[**Hint.** Proceed exactly as in Solved Example 12.
Given $p_1 = .3$, $p_2 = .7$, $p_3 = 0$. We find $N_1 = 30$, $N_2 = 79$. Expected life = 1.7. Average number of failing per week 100/1.7 = 59. Average costs at the end of each week will be 87.50, 93.12, 93.87 respectively.]
[**Ans.** It is optimal to have a group replacement after every second week. But, the pure individual replacement costs Rs. 59 × 1.25 = Rs. 73.75. Hence the **individual replacement** is preferable.]

4. Let $p(t)$ be the probability that a machine in a group of 30 machines would break down in period t. The cost of repairing broken machine is Rs. 200.00. Preventive maintenance is performed by servicing all the 30 machines at the end of T units of time. Preventive maintenance cost is Rs. 15 per machine. Find optimal T which will minimize the expected total cost per period of servicing, given that

$$p(t) = \begin{cases} 0.03 & \text{for } t = 1 \\ p(t-1) + 0.01 & \text{for } t = 2, 3, \ldots, 10 \\ 0.13 & \text{for } t = 11, 12, 13, \ldots \end{cases}$$

[JNTU (Mech. & Prod.) 2004; Meerut (M.Sc. Maths) 90]

[**Hint.** Here,

$t = 1$	2	3	4	5	6	7	8	9	10	11	12	13
$p_t = 0.03$	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0	0

Since the sum of all probabilities can never be greater than 1, all the probabilities beyond p_{11} will be taken zero. This means that a machine which has already lasted up to 11th period is sure to fail in the 12th period. Also, we have
$N_0 = 30$, $N_1 = 1$, $N_2 = 1$, $N_4 = 2$, $N_5 = 2$, $N_6 = 3$, etc.
Average costs are : Rs. 650, 425, 416, 412, 410, 442.]
[**Ans.** It is optimal to maintain all the machines at the end of 5th period. This will cost Rs. 41.00.]

5. There is a large number of light bulbs, all of which must be kept in working order. If a bulb fails in service, it costs Re. 1 to replace it, but if all the bulbs are replaced in the same operation, it costs only 35 paise a bulb. If the proportion of bulbs failing in succesive time intervals is known, decide on the best replacement policy and give resons. The following mortality rates for light bulbs have been observed. :

Proportion failing during first week = 0.09
" " " second " = 0.16
" " " third " = 0.24
" " " fourth " = 0.34
" " " fifth " = 0.12
" " " sixth " = 0.03

[**Hint.** Assume $N_0 = 1000$. Here $p_1 = .09$, $p_2 = .16$, $p_3 = .24$, $p_4 = .36$, $p_5 = .12$, $p_6 = .03$. Expected life $\sum_{i=1}^{6} i p_i = 3.35$.

Average number of failure per week = 1000/3.35 = 299 (nearly) Cost of individual replacement = Rs. 299.
Compute group replacement cost at the end of each week.]
[**Ans.** It is optimal to replace all bulbs at the end of third week.]

6. An electric company which generates and distributes electricity conducted a study on the life of poles. The appropriate life data are given in the following table :

Years after installation	:	1	2	3	4	5	6	7	8	9	10
Percentage poles failing	:	1	2	3	5	7	12	20	30	16	4

(*i*) If the company now installs 5,000 poles and follows a policy of replacing poles only when they fail, how many poles are expected to be replaced each year during the next ten years.

To simplify the computation assume that failures occur and replacements are made only at the end of a year.

(*ii*) If the cost of replacing individually is Rs. 160 per pole and if we have a common group replacement policy, it costs Rs. 80 per pole, find out the optimal period for group replacement.

7. It has been suggested by a data processing firm that they adopt a policy of periodically replacing all the 1000 tubes in a certain piece of equipment. A given type of tube is known to have the mortality distribution (prob. of failure) shown in the table below :

Tube failures/week	:	1	2	3	4	5
Prob. of failure	:	0.3	0.1	0.1	0.2	0.3

The cost of replacing the tubes on an individual basis is estimated to be Re. 1.00 per tube and the cost of a group replacement policy average Re. 0.30 per tube. Compare the cost of preventive replacement with that of remedial replacement. **[Raj. Univ. (M. Phil) 91]**

[**Ans.** It is optimal to have a group replacement after every fourth week along with the individual replacement.]

8. A factory has a large number of bulbs, all of which must be in working condition. The mortality of bulbs is given in the following table :

Week	:	1	2	3	4	5	6
Proportion of bulbs failing during	:	0.10	0.15	0.25	0.35	0.12	0.03

If a bulb fails in service, it costs Rs. 3.50 to replace; but if all the bulbs are replaced at a time it costs Rs. 1.20 each. Find the optimum group replacement policy.
[**Ans.** After every 3rd week.]

9. Find the cost per period of individual replacement policy of an installation of 300 lighting bulbs, given the following :

(*i*) cost of replacing individual bulb is Rs. 3,

(*ii*) conditional probability of failure is given below :

Week no.	:	0	1	2	3	4
Conditional prob. of failure	:	0	1/10	1/3	2/3	1

[**Ans.** The expected life of each light bulb is 3 weeks and the average cost of individual replacement of 300 light bulbs is Rs. 206.]

10. A decorative series lamp set circuit contains 10,000 bulbs, when any bulb fails, it is replaced. The cost of replacing a bulb individually is Re. 1 only. If all the bulbs are replaced at the same time, the cost per bulb would be reduced to 35 paise. The per cent surviving, say $S(t)$, at the end of month t and, $P(t)$, the probabilities of failure during the month, are given below :

t	0	1	2	3	4	5	6
$S(t)$	100	97	90	70	30	15	0
$P(t)$	–	0.03	0.07	0.20	0.40	0.15	0.15

What is the optimal replacement ? **[VTU (BE Mech.) 2002]**
[**Ans.** It is optimal to have a policy of group replacement after every third month.]

11. There are 1000 bulbs in the system. Survival rate is given below :

Week	:	0	1	2	3	4
Bulbs in operation of the end of week	:	1000	850	500	200	00

The group replacement of 100 bulbs costs Rs. 1000 and individual replacement is Rs. 20 per bulb. Suggest suitable replacement policy. **[JNTU (B. Tech.) 2003]**

12. There is a special light bulb that never lasts longer than 2 weeks. There is a chance of 0.3 that a bulb will fail at the end of first week. There are 100 new bulbs initially. The cost for individual replecement is Rs. 1.25 and cost per bulb for group replacement is Rs. 0.50. Is it cheaper to replace all the bulbs, (i) individually (ii) every week (iii) every second week.
[JNTU (Mech. & Prod.) 2003]

13. A large computer has 2000 components of identical nature which are subjected to falure as per the probability distribution given below :

Week end	:	1	2	3	4	5
Probability of falure	:	0.10	0.25	0.50	0.80	1.00

If the cost of individual replacement per unit is Rs. 3 and for group replacement per unit is Re 1, assess which of the replacement would be economical and when ? **[JNTU (Mech. & Prod.) May 2004]**

III – Other Replacement Problems

There are many other replacement situations which have not been classified so for. Some of them are discussed in the following sections.

22.14. RECRUITMENT AND PROMOTION PROBLEMS

Like industrial replacement problems, principles of replacement are also applicable to the problems of recruitment and staff promotion. In staffing problems, with fixed total staff and fixed size of staff groups, the proportion of staff in each group determines the promotion age, and conversely. Unemployment situations in these days can often be considerably improved by the possibility of expansion. Following example will make the principles clear.

Example 17. (***Staffing Problems***). *An airline requires 200 assistant hostesses, 300 hostesses, and 50 supervisors. Girls are recruited at the age of 21, if still in service retire at age 60. Given the following Life Table, determine*

(*i*) *How many girls should be recruited in each year ?*

(*ii*) *At what age promotion should take place.*

Table 22.11. Airline Hostesses Life Record

Age	21	22	23	24	25	26	27	28
No. in Service	1000	600	480	384	307	261	228	206
Age	29	30	31	32	33	34	35	36
No. in Service	190	181	173	167	161	155	150	146
Age	37	38	39	40	41	42	43	44
No. in Service	141	136	131	125	119	113	106	99
Age	45	46	47	48	49	50	51	52
No. in Service	93	87	80	73	66	59	53	46
Age	53	54	55	56	57	57	59	–
No. in Service	39	33	27	22	18	14	11	–

Solution. The total number of girls recruited at the age of 21 and those serving up to the age of 59 is 6480. In all, 200 + 300 + 50 = 550 girls are required in the airline.

Every year recruitement is 1000 when total number of girls are 6480 up to the age of 59 years. Therefore, in order to maintain a strength of 550 hostesses, recruitement policy should be $550 \times 1000/6480 = 85$ (nearly) every year.

If promotion is given to the assistant hostesses at the age x, then up to age $x - 1$, 200 assistant hostesses will be required. Among 550, there are 200 assistant hostesses. Therefore, out of a strength of 1000, there will be

$$200 \times 1000/550 = 364 \text{ assistant hostesses,}$$

and from the *Life Table*, this number is available up to the age of 24 years. Hence the promotion of assistant hostesses will be due in 25th year.

Also out of 550 recruitments, only 300 hostesses are needed.

Therefore, if 1000 hostesses are recruited, then only $\left(\frac{1000 \times 300}{550} \sim 545\right)$ hostesses are required.

Hence, the number of hostesses and assistant hostesses in a recruitment of 1000 will be $= 364 + 545 = 909$.

So, only $1000 - 909 = 91$ supervisors are needed whereas at the age of 46 only 87 supervisors will survive. Hence, promotion of hostesses to supervisors will be due in 46th year.

Example 18. *Calculate the probability of a staff resignation in each year from the following survival table :*

Year	:	0	1	2	3	4	5	6	7	8	9	10
No. of original staff in service at the end of year	:	1000	940	820	580	400	280	190	130	70	30	0

[Raj. univ. (M. Phil) 93]

Solution. Let p_i denote the probability of a staff resignation (failure) in the ith year. Thus, we have

$$p_i = \frac{N(i-1) - N(i)}{N},$$

Where N is the total number of staff members in the system and $N(i)$ is the total number of staff members in the system at the end of ith year.

It is given that

$N(0) = 1{,}000,\ N(1) = 940,\ N(2) = 820,\ N(3) = 580,\ N(4) = 400,\ N(5) = 280,$
$N(6) = 190,\ N(7) = 130,\ N(8) = 70,\ N(9) = 30, \text{ and } N(10) = 0.$

The required probability of staff resignation in each year can be calculated as below :

$$p_1 = \frac{N(0) - N(1)}{N} = \frac{1000 - 940}{1000} = 0.06, \qquad p_6 = \frac{N(5) - N(6)}{N} = \frac{280 - 190}{1000} = 0.09$$

$$p_2 = \frac{N(1) - N(2)}{N} = \frac{940 - 820}{1000} = 0.12 \qquad p_7 = \frac{N(6) - N(7)}{N} = \frac{190 - 130}{1000} = 0.06$$

$$p_3 = \frac{N(2) - N(3)}{N} = \frac{820 - 580}{1000} = 0.24 \qquad p_8 = \frac{N(7) - N(8)}{N} = \frac{130 - 70}{1000} = 0.06$$

$$p_4 = \frac{N(3) - N(4)}{N} = \frac{580 - 400}{1000} = 0.18 \qquad p_9 = \frac{N(8) - N(9)}{N} = \frac{70 - 30}{1000} = 0.04$$

$$p_5 = \frac{N(4) - N(5)}{N} = \frac{400 - 280}{1000} = 0.12 \qquad p_{10} = \frac{N(9) - N(10)}{N} = \frac{30 - 0}{1000} = 0.03$$

Example 19. *A research team is planned to raise to a strength of 50 chemists and then to remain at that level. The wastage of recruits depends on their length of service which is as follows :*

Year	:	1	2	3	4	5	6	7	8	9	10
Total % who have left upon the end of year	:	5	36	56	63	68	73	79	87	97	100

What is the recruitment per year necessary to maintain the required strength ? There are 8 senior posts for which the length of service is the main criterion. What is the average length of service after which new entrant expects promotion to one of the posts ? **[Rohilkhand 91]**

Solution. Since the % of chemists who have left upto the end of the year, the probability p_i of a person leaving during the ith year can be determined.

The required probability of a person being in service at the end of the year is calculated in the following table :

Year (n)	No. of persons who left at the end of year	No. of persons in service at the end of year	Prob. of leaving at the end of year	Prob. of persons in service at the end of year (p_n)
0	0	100	0	1.00
1	5	95	0.05	0.95
2	36	64	0.36	0.64
3	56	44	0.56	0.44
4	63	37	0.63	0.37
5	68	32	0.68	0.32
6	73	27	0.73	0.27
7	79	21	0.79	0.21
8	87	13	0.87	0.13
9	97	3	0.97	0.03
10	100	0	1.00	0

Above table shows that if 100 chemists are appointed each year then the total number of chemists serving in the organization would have been 436.

Thus, in order to maintain a strength of 50 chemists, we must recruit,

$$\frac{100 \times 50}{436} = 12 \text{ chemists every year.}$$

If p_n denotes the probability of a person to be in service at the end of n^{th} year, then out of 12 recruits the total number of survivals at the end of year n will be $12 \times p_n$. Thus we can construct the following table showing the number of chemists in service at the end of each year.

Years (n)	:	0	1	2	3	4	5	6	7	8	9	10
Prob. p_n	:	1.00	0.95	0.64	0.44	0.37	0.32	0.27	0.21	0.13	0.03	0
No. of Chemists ($12 \times p_n$)	:	12	11	8	5	4	4	3	2	2	0	0

There are 8 senior posts for which the length of service is the main criterion.

From the above table, we observe that there are 3 persons in service during the 6th year, 2 persons in 7th year and 2 persons in 8th year, *i.e.*, in all 7 persons are there in service from 6th to 8th year which is less than the total number of senior posts.

Hence the promotions of the new entrants will start by the end of 5th year.

Example 20. *An airline, whose staff members are subject to the same survival rates as is in the previous problem, presently has a staff whose ages are distributed in the following table. It is estimated that for the next two years staff requirements will increase by 10% per year. If girls are to be recruited at the age of 21, how many should be recruited for next year and at what age will promotions take place ? How many should be recruited for the following year and at what age will promotions take place ?*

Assistant :

Age	:	21	22	23	24	25	
Number	:	90	50	30	20	10	(Total 200)

Hostesses :

Age	:	26	27	28	29	30	31	32	33	34
Number	:	40	35	35	30	28	26	20	18	16
Age	:	35	36	37	38	39	40	41		
Number	:	12	10	8	–	8	8	6	(Total 300)	

Supervisors :

Age	:	42	43	44	45	46	47	48	49	50	
Number	:	5	4	5	3	3	3	6	2	–	
Age	:	51	52	53	54	55	56	57	58	59	
Number	:	–	4	3	5	–	3	2	–	2	(Total 50)

Solution. Compute the expected number of survivals from supervisors for one year by making use of the probability of survival for one year at each age as shown in the following table :

Present Age	No. of supervisors	Prob. of being in service for one year hence	Expected survivals after one year	New age
(1)	(2)	(3)	(4)	(5)
42	5	0.947	4.735	43
43	4	0.942	3.768	44
44	5	0.936	4.680	45
45	3	0.923	2.790	46
46	3	0.915	2.769	47
47	3	0.906	2.745	48
48	6	0.906	5.436	49
49	2	0.896	1.92	50
50	0	0.885	0	51
51	0	0.873	0	52
52	4	0.860	3.440	53
53	3	0.846	2.538	54
54	5	0.831	4.155	55
55	0	0.815	0	56
56	3	0.798	2.394	57
57	2	0.780	1.560	58
58	0	0.761	0	59
59	2	0.741	1.482	60
	50		**43**	

From this table, we observe that only 43 supervisors will be in service after one year. Also, there is 10% increase in the posts, *i.e.*, there will be 55 posts of supervisors after one year and only 43 will remain in service. Hence 12 hostesses are to be promoted on the basis of their age.

Since there are 6 hostesses of age 41 and their probability of survival for one year is 0.952, expected number of hostesses aged 41 who will be in service for one year more (*i.e.*, up to age of 42 years) $= 6 \times 0.952 = 5.712$.

Similarly, there are 8 hostesses of age 40 years with probability of survival for one year being 0.957, the expected numbr of hostesses aged 40 who will be in service for one year more = 8 × 0.957 = 7.656.

Thus after one year from present age, (5.712 + 7.656) = 13.3 seniormost hostesses will be of age 42 and 41 years and we want to promote 12 out of them. Hence all hostesses of age 41 and all but one of age 40 will be promoted as supervisors after one year.

Remark. The values in column (3) of above table can be computed from the data of given ***example, e. g.***

$$0.947 = \frac{\text{No. in service at age 42}}{\text{No. in service at age 41}} = \frac{113}{119}, \text{ and so on.}$$

Q. Write short note on staffing problem. **[Garhwal M.Sc. (Stat.) 94]**

22.15. EQUIPMENT RENEWAL PROBLEM

The term '*renewal*' means that either to *insert* a new equipment in place of an old one or to *repair* the old equipment so that the probability density function (*p.d.f*) of its future life time is that of new equipment. Here life-span of the equipment is considered to be a random variable.

Renewal Rate. ***Definition*** *. The probability that a renewal occurs during the small interval $(t, t + \delta t)$ is called the renewal rate at time t, where t is measured from the instant the first machine was started. It is denoted by $h(t)dt$ and also called the* ***renewal density function.***

Theorem 22.5. *The renewal rate of a machine is asymptotically reciprocal of the mean life of the machine.*

Proof. Let $f(x)$ be the *p.d.f.* of failure time of a machine. If $X_i (i = 1, 2, 3, ...)$ is life-span for ith machine, then each $X_1, X_2 \ldots$ also has the *p.d.f.* of $f(x)$. Also, let the machine fail $(n - 1)$ number of times during period t and as soon as the machine fails it is immediately replaced by a similar other machine. Suppose nth machine is in service at the end of this period. Then,

$$X_1 + X_2 + \ldots + X_{n-1} < t \quad \text{and} \quad X_1 + X_2 + \ldots + X_n > t.$$

For convenience, denote $$P\left(t \le \sum_{i=1}^{r} X_i \le t + dt\right) = f_r(t)\, dt.$$

Since rth machine can fail during period $(t, t + \delta t)$, Therefore

$$h(t) = \sum_{r=1}^{\infty} f_r(t),$$

Therefore, if N machines are used at a time, the expected number of replacements at the end of tth period is

$$= N \int_0^t h(y)\, dy.$$

It is not necessary that $\int_0^\infty h(t)\, dt = 1$, because $h(t)$ is not the *p.d.f.* But $h(t)$ possesses the additive property,

$$h(t_2 - t_1) = h(t_2) - h(t_1), \; t_2 > t_1.$$

Now to prove this theorem, the *Laplace-Stielzes* transforms of functions $h(t)$ and $f(t)$ are defined as

$$h^*(z) = \int_0^\infty e^{-zt} h(t)\, dt \text{ and } f^*(z) = \int_0^\infty e^{-zt} f(t)\, dt.$$

Thus, $$h^*(z) = \int_0^\infty e^{-zt}\left[\sum_{r=1}^{\infty} f_r(t)\right] dt = \sum_{r=1}^{\infty}\left[\int_0^\infty e^{-zt} f_r(t)\, dt\right] = \sum_{r=1}^{\infty} [f^*(z)]^r$$

$[Sf_r^*(z) = \int_0^\infty e^{-zt} f_r(t)\, dt = [f^*(z)]^r$, by using the well known property of *Laplace-Stieltzes transform*]

$$\therefore \quad h^*(z) = f^*(z)/[1 - f^*(z)] \qquad \text{(sum of infinite G.P.)}$$

If the mean life of the machine be given by $\lambda = \int_0^\infty x f(x)\, dx$, then for $r \to 0$,

$$h^*(z) \to \frac{1 - \lambda z + \ldots}{\lambda z - \ldots} \to 1/\lambda z \text{ (neglecting higher powers of z)}$$

Taking inverse Laplace transform of both sides,

$$h(t) \to 1/\lambda \text{ (reciprocal of mean-life of machine)}$$

This completes the proof of the theorem.

Q. 1. What is renewal function ? Suppose the life of electric light bulbs follows the following distribution :

$$f(x)dx = \lambda x^{-\lambda x}\, dx \quad (\lambda > 0;\ 0 \le x < \infty)$$

Determine the renewal density $h(t)$ after the end of the period $(0, t)$.

If there are N points in a house, how many bulbs would you expect to be replaced within a period (t_1, t_2) where $t_2 > t_1$?

[Raj. univ. (M. phil) 91, 90]

2. Describe the various replacement models. **[Garhwal M.Sc. (Stat.) 96, 95, 93]**
3. Define renewal function. What is renewal theory ? Derive the Fundamental Integral Equation of the renewal theory for an ordinary renewal process. **[Raj Univ. (M. Phil) 93, 92]**

22.15 – I. Illustrative Examples

Example 21. *Suppose the life X of electric light bulb follows the gamma distribution :*

$$P(x \le X < x + dx) = \frac{a^p}{\Gamma(p)} e^{-ax} x^{p-1}\, dx \quad (a > 0, 0 \le x < \infty)$$

Determine the renewal rate for one point at the end of time period $(0, t)$.

Solution. By virtue of the additive property of *gamma distribution*

$$f_r(x)\, dx = P\left(x \le \sum_{i=1}^{r} X_i < x + dx\right) = \frac{a^{rp}}{\Gamma(rp)} e^{-ax} x^{rp-1} dx$$

Hence by definition of renewal rate

$$h(t) = e^{-ax} \sum_{r=1}^{\infty} \frac{a^{rp} x^{rp-1}}{\Gamma(rp)} \to \sum_{r=1}^{\infty} \frac{a^{rp} x^{rp-1}}{\Gamma(rp)} \quad \text{as } t \to \infty.$$

Example 22. *A piece of equipment can either fail completely so that it has to be scraped (no salvage value), or may suffer a minor defect which can be repaired. The probability that it will not have to be scraped before age t is f(t). The conditional probability that it will need a repair in the interval t to t + dt, given that it is in running order at age t, is r(t)dt. Probability of a repair or complete failure is dependent only on the age of the equipment and not on the previous repair history.*

The repair cost is C and complete replacement cost is K. For some considerable time, the policy has been to replace only on failure.

(*a*) *Derive formula for the expected cost per unit time of the present policy of replacing only on failure.*

(*b*) *It has been suggested that it might be cheaper to scrap equipment at some fixed age T, thus avoiding the higher risk of repair with advancing age. Show that the expected cost per unit time of such a policy is*

$$\left[C \int_0^T f(u)\, r(u)\, du + K\right] \Big/ \int_0^T f(u)\, du.$$

(*c*) *By differentiating the expression, find a condition to be satisfied by T for minimum cost.*

[Meerut M.Sc. (Math.) BP-96]

Solution. (*a*) Since $f(t)$ is the probability density when equipment is not scraped before age t, therefore

$$\int_0^\infty f(t)\, dt = 1.$$

Also, the probability that equipment will need repairing in interval $(t, t + dt)$ when the equipment was running at age t is $r(t)\, dt$. But the probability that equipment will need repairing between the ages u and $u + du$ is $f(u)du$.

Now, expected cost of repairing becomes $= C \int_0^\infty f(u)\, r(u)\, du$,

and hence the total expected cost becomes

$$\left[K + C \int_0^\infty f(u)\, r(u)\, du\right] \Big/ \int_0^\infty f(u)\, du. \; = K + C \int_0^\infty f(u)\, r(u)\, du.$$

(*b*) Considering the policy of scrapping at age T, the expected cost of repair becomes

$$= \int_0^T f(u)\, r(u)\, du$$

and hence the total expected cost up to age T becomes

$$= K + C \int_0^T f(u)\, r(u)\, du\,.$$

Thus, the expected cost per unit time is given by

$$E(T) = \left[K + C \int_0^T f(u)\, r(u)\, du \right] \Big/ \int_0^T f(u)\, du.$$

(*c*) Finally, obtain the value of T for which $E(T)$ is minimum. The necessary condition for minimization is

$$\partial E(T)/\partial T = 0.$$

or $$\frac{C f(T)\, r(T) \int_0^T f(u)\, du - \left[K + C \int_0^T f(u)\, r(u)\, du \right] f(T)}{\left[\int_0^T f(u)\, du \right]^2} = 0$$

or $$r(T)\, C \int_0^T f(u)\, du = K + C \int_0^T f(u)\, r(u)\, du$$

or $$\int_0^T f(u)\, [r(T) - r(u)]\, du = \frac{K}{C}.$$

But, $$K = E(T) \int_0^T f(u)\, du - C \int_0^T f(u)\, r(u)\, du.$$

Therefore, $$\int_0^T f(u)\, [r(T) - r(u)]\, du = \frac{E(T)}{C} \int_0^T f(u)\, du - \int_0^T f(u)\, r(u)\, du$$

or $$r(T) \int_0^T f(u)\, du = \frac{E(T)}{C} \int_0^T f(u)\, du$$

or $$E(T) = r(T)\, C.$$

Thus, minimum value of $E(T)$ is equal to $r(T)\, C$.

Example 23. *A certain piece of equipment is extremely difficult to adjust. During a period when no adjustment is made, the running cost increases linearly with time at a rate of b rupees per hour. The running cost immediately after an adjustment is not known precisely until the adjustment has been made. Before the adjustment, the resulting cost x is deemed to be a random variable x with density function f(x). If each adjustment costs k rupees, when should replacement be made ?* **[Meerut (OR) 2003]**

Solution. Let X be the maximum value of random variable x, *i.e.*,

$$0 \le x \le X.$$

If the adjustment is done only when the running cost becomes z (say), then only two possibilities arise :

(*i*) $z > X$, and (*ii*) $z < X$.

Case 1. When $z > X$. Let x be the running cost at time $t = 0$ and the adjustment be after time t. Then the running cost at the time of adjustment 't' becomes Rs. $(x + bt)$, *i.e.*

$$z = x + bt \quad \text{or} \quad t = (z - x)/b.$$

If $C_1(z, x)$ denotes the total cost during the period of one adjustment, then

$C_1(z, x)$ = total running cost from $\{t = 0$ to $t = (z - x)/b\}$ + the cost of adjustment

Fig. 22.1. $f(x)$ **for running cost after an adjustment.**

$$= \int_0^{(z-x)/b} (x + bt)\, dt + k = \frac{1}{2b}(z^2 - x^2) + k \quad ...(22.37)$$

Hence the average cost per hour is

$$A_1(z, x) = \frac{C_1(z, x)}{\text{cycle time}} = \frac{(z^2 - x^2)/2b + k}{(z - x)/b} = \frac{z + x}{2} + \frac{bk}{z - x}.$$

Since $f(x)$ is the *p.d.f.* for x, the expected cost per hour becomes

$$E_1(z, x) = \int_0^X \left(\frac{z + x}{2} + \frac{bk}{z - x} \right) f(x)\, dx \quad ...(22.38)$$

To minimize E_1, differentiate w.r.t. 'z' under the sign of integration [see page 37, ***Unit*-1**] and get

$$\frac{dE_1}{dz} = \frac{\partial}{\partial z} \int_0^X \left(\frac{z + x}{2} + \frac{kb}{z - x} \right) f(x)\, dx$$

$$= \int_0^X \left(\frac{1}{2} - \frac{kb}{(z - x)^2} \right) f(x)\, dx$$

$$= \tfrac{1}{2} \int_0^X f(x)\, dx - kb \int_0^X \frac{f(x)}{(z - x)^2}\, dx$$

$$= \tfrac{1}{2} - kb \int_0^X \frac{f(x)}{(z - x)^2}\, dx \quad \left(\because \int_0^X f(x)\, dx = 1 \right)$$

Hence, for $E_1(z, x)$ to be minimum,

$$\frac{dE_1}{dz} = \frac{1}{2} - kb \int_0^X \frac{f(x)}{(z - x)^2}\, dx = 0$$

which gives

$$\int_0^X \frac{f(x)}{(z - x)^2}\, dx = \frac{1}{2kb}. \quad ...(22.39)$$

This formula can yield the best value z_0 of z, at least in the range $z > X$.

Case 2. When $z < X$. In this situation, repeated adjustments are necessary at time $t = 0$ before the beginning of the cycle. If the running cost is x at the end of previous cycle, *i.e.*, just before $t = 0$, then the probability that no further adjustment is required at $t = 0$, is equal to the probability of the event $\{x < z\}$.

For simplicity, use the notation $\int_0^X f(x)\, dx = F(z)$, (say)

to obtain $\quad P\{\text{no further adjustment at } t = 0\} = 1 - F(z) \quad ...(22.40)$

Also, it can be observed that

$$P\{r \text{ further adjustments at } t = 0\} = r[1 - F(z)]\,[F(z)]^r.$$

Since the cost of each such adjustment is Rs. k, the expected cost of these further adjustments at $t = 0$ is

$$\sum_{r=1}^{\infty} kr\,[1 - F(z)]\,[F(z)]^r = k\,[1 - F(z)] \sum_{r=1}^{\infty} r[F(z)]^r = k\,[F(z)/[1 - F(z)]\dagger$$

After making necessary adjustments with running cost x ($< z$), the remainder of the cycle cost will be same as in equation (22.37). Thus, adding (22.37) to the above expected adjustment cost that yields the cycle cost,

$$C_2(z, x) = \frac{z^2 - x^2}{2b} + k + \frac{kF(z)}{1 - F(z)} = \frac{z^2 - x^2}{2b} + \frac{k}{1 - F(z)}$$

---

† $\sum_{r=1}^{\infty} r[F(z)]^r = F(z) + 2[F(z)]^2 + 3[F(z)]^3 + \dots = S$ (*say*)

$\therefore$ Subtracting $\quad [F(z)]^2 + 2[F(z)]^3 + \dots = S\,F(z)$

$\quad F(z) + [F(z)]^2 + [F(z)]^3 + \dots = S[1 - F(z)]$

or $\quad F(z)/[1 - F(z)] = S[1 - F(z)]$

$\therefore \quad S = F(z)/[1 - F(z)]^2$

Cycle time still being $(z-x)/b$, the average cost per hour is

$$\frac{C_2(z, x)}{\text{cycle time}} = \frac{z+x}{2} + \frac{kb}{(z-x)[1-F(z)]} = A_2(z, x), \text{ (say)}$$

To calculate the expected cost per hour, the density function is conditional rather than $f(x)$, *i.e.*

$$f(x \mid x < z) = \frac{f(x)}{P[\text{no further adjustment at } t = o]} = \frac{f(x)}{1-F(z)} \qquad \text{[from eqn. (22.40)]}$$

Hence, the expected cost per hour is

$$E_2(z) = \int_0^z f(x \mid x < z)\, A_2(z, x)\, dx$$

$$= \int_0^z \frac{f(x)}{1-F(z)} \left[\frac{z+x}{2} + \frac{kb}{(z-x)[1-F(z)]}\right] dx$$

$$= \frac{1}{2[1-F(z)]} \int_0^z (z+x) f(x)\, dx + \frac{kb}{[1-F(z)]^2} \int_0^z \frac{f(x)}{z-x}\, dx.$$

Since the second integral involving $f(x)/(z-x)$ is of infinite order, are cannot obtain the minimum value of $E_2(z)$. Hence, the optimum value z_0 of z, can be obtained in previous case only (when $z < X$) in which $E_1(z_0)$ was obviously finite.

Example 24. *A certain piece of equipment is extremely difficult to adjust. During a period when no adjustment is made the running cost increases as at + b (where* a *and* b *are constants). The running cost immediately after the adjustment is not known until the adjustment has been made. After the adjustment, the resulting running cost x is a random variable with prob. density function f (x) is given by* $f(x) = px^2 + q$*. If each adjustment costs K rupees, when should adjustment be made ?* **[Meerut (Maths.) Jan. 98, BP]**

[**Hint** : Proceed similarly as Ex. 23]

IV— Systems Reliability

22.16. DEFINITIONS

The concept of a system may be easily understood by following definitions :

Definition 1. *A system is a set of units with relationship among them.*

Definition 2. *A system is an aggregation or assemblage or objects united by some form of regular interaction of interdependence; a group of diverse units so combined by nature or art as to form an integral whole and to function, operate or move in unison and often in obedience to some form of control.*

Definition 3. *A system consisting of a set of entities among which a set of relationship is specified so that deductions are possible from one relation to other or form relations among entities to the behaviour or the history of the system.*

Definition 4. *A system is an assembly of procedures, processes, methods, routines or techniques united by some form of regulated interaction to form an organized whole.*

Definition 5. *In general, a system is a collection of interacting sub-systems.*

The following are some examples of the system :

(*i*) Ministry of Education : *System,* (*ii*) University Grants Commission : *Sub-system.*

(*iii*) University of Meerut : Sub-sub-system. (*iv*) South Campus : Sub-sub-sub-system.

(*v*) Institute of Advanced Studies : Sub-sub-sub-sub-system.

(*vi*) Department of Operations Research : Sub-sub-sub-sub-sub-system.

22.17. CLASSIFICATION OF SYSTEMS

Systems can be generally classified into two main heads, namely,

Physical systems and socio-economic systems. Some typical examples of systems are :

(1) Atomic system	(2) Banking system	(3) Data processing system
(4) Educational system	(5) Heating system	(6) Law and order system
(7) Legal system	(8) Panel system	(9) Pay roll system

(10) Social system (11) Transportation system (12) Telephone system
(13) Value system (14) Weapons system (15) Weather system.

22.18. ENVIRONMENTAL FACTORS OF SYSTEM

Environments are dynamic and hence are likely to change over the life of the system and thus may cause failure or breakdown of the system as a whole. Such environmental factors are :

(*a*) Economic environment (*b*) Organizational environment
(*c*) Political environment (*d*) Time environment
(*e*) Human environment (*f*) Physical environment
(*g*) Technological environment.

Closed system. *A system where no interaction with the environment is considered **is called a closed system.***

Open system. *A system where all possible effects are considered of environments on the systems and vice-versa is called an **open system.***

22.19. CONCEPT OF SYSTEM RELIABILITY

The system reliability is important because almost all physical and a lot of abstract conceptual systems may breakdown after sometime.

A breakdown of the system involves an actual failure as well as unsatisfactory performance or malfunctioning. Thus, a system is capable of giving performance over a longer period is said to be more reliable.

Definitions of Reliability :

(*a*) Reliability is the probability of a device performing its purpose adequately for the period of time intended under operating conditions encountered.

(*b*) Probability of trouble free operation throughout a specified period of time.

Some important characteristics of reliability are :

(*i*) Reliability is expressed as a probability which helps to quantify and think of optimizing the system reliabililty.

(*ii*) Reliability is a function of time. Almost burnt-out light bulbs are not expected to be as reliable as one recently put into service.

(*iii*) Reliability is a function of conditions of use. In every severe environments, frequent system breakdowns than in normal environments are encountered.

22.20. MATHEMATICAL MODEL OF SYSTEM RELIABILITY

If the event A is the survival of the system and event B is the system failure *(breakdown)*, then the reliability of the system can be expressed as the ratio of number of systems surviving the test to the number of systems present and operating at the beginning of the test.

Let N be the units to start with. After a time t, let the number that survived the test be $N_s(t)$ and number that fail is $N_f(t)$.

Then, $R(t) = N_s(t)/N = 1 - [N_f(t)/N]$, for fixed N.

Therefore, $$\frac{dR(t)}{dt} = -\frac{1}{N}\frac{dN_f(t)}{dt} \quad \text{or} \quad \frac{dN_f(t)}{dt} = -N\frac{dR(t)}{dt}$$

where the function $dN_f(t)$ denotes the rate at which components fail.

For random situations, failures follow the *Poisson distribution*

$$P_n(t) = e^{-\lambda t}(\lambda t)^n/n!,$$

where $P_n(t)$ is the probability of exactly n failures in times t when the average failure rate is λ and therefore

$R(t) = P_0(t) =$ Prob. of no failure during period $t = e^{-\lambda t}$

Hence, the reliability function for random failures is an *exponential distribution.*

22.21. IMPROVEMENT OF RELIABILITY

The system reliability can be improved either by providing redundance in the system or by improving the design. The system with redundance has a number of standby units which takeover if the other component of the system fails. Thus, a reserve stock of standby improves the system reliability but at a higher cost. A compromise between the increased cost due to standby arrangement and increased given cost of reliability decides the optimal redundancy in the system.

22.22. MAINTAINABILITY OF THE SYSTEM

Maintainability of the system is another property which is defined as the probability that a system, if failed, will be brought back to operational effectiveness within a prescribed period of time if maintenance action is carried out in a prescribed manner. With the help of this maintainability concept, one can bring the system back under control. In case, maintainability and reliability of a system is high, it will have a high availability in the sense that most of the time the system will remain under control.

Q. **1.** Discuss in brief : (*i*) System Reliability (*ii*) System Maintainability. **[Raj. Univ. (M.Phil) 90]**

2. Define : (*i*) Reliability (*ii*) Interval availability (iii) Limiting interval availability.

3. Explain the difference between Reliability and Point Availability.

SELF-EXAMINATION QUESTIONS

1. Write a critical essay on replacement problems.
2. Discuss briefly the various types of replacement problems.
3. (i) Write a short note on replacement problems.
   (ii) Discuss the importance of replacement models.
4. Give a brief account of situations of which the replacement problems arise. What does the theory of replacement establish ?
5. State some of the simple replacement policies and given the average cost functions for the same, explaining notations used.
6. Discuss the brief replacement procedure for the items that deteriorate with time.
7. Discuss the various methods of depreciating an asset.
8. Explain how one can compare two or more replacement policies. Compare the following replacement policies :
   (i) Replacement on failure at a cost c.
   (ii) Replacement on the attainment of age t with cost c_2 ($c_2 > 0$) or on failure, if it happens earlier at a cost c_1 ($> c_2$).
9. Consider a series of periods 1, 2, 3, ... of equal length and let the costs incurred on a certain equipment in these periods be $c_1, c_2, \ldots$, respectively. Suppose that each cost is paid at the beginning of the period in which it is incurred, and that the worth of money is 10%. Determine the optimum policy for the replacement of the equipment.
10. Discuss the treatment of unequal service life in replacement analysis. Discuss the minimum cost replacement model. Under what conditions is the minimum cost replacement model applicable ?
11. Discuss in detail "individual replacement policy", "group replacement policy", and "replacement policy with maintenance costs increasing with time.
12. State some of the simple replacement policies.
13. Write a short note on considerations leading to equipment replacement.
14. Explain how the theory of replacement is used in the following problems :
    (*i*) Replacement of items whose maintenance cost varies with time.
    (*ii*) Replacement of items that fail completely.
15. A company runs a machine shop containing an expensive drill press that must be replaced periodically as it wears out. The Vice-President of the company has just authorized the installing of a new model but has requested you to devise an optimal replacement plan for next seven years, after which the drill press will not be needed. State what information you will need and how you will use it.
    [Your answer should include the mathematical solution to the problem and indicate the reservation, if any, in the use of this solution.]
16. It is required to find the optimum replacement time of a certain type of euipment. The initial cost of equipment is C. salvage value and repair costs are given by $S(t)$ and $R(t)$, respectively. The cost of capital is r per cent and T is the time period of replacement cycle.
    (*i*) Show that the present value of all future costs associated with a policy of equipment after T is
    $$\left(\frac{1}{1'- e^{-rt}}\right) \left[C - S(t)\, e^{-rt} + \int_0^T R(t)\, e^{-rt}\, dt\right].$$
    (*ii*) The optimal value of T is given by
    $$R(t) - S'(T) + S(t)\, r = r^k/(1 - e^{-rt})$$
    where k is the present value of the cycle.

17. There are N lamps on a life test and the testing is terminated as soon as the rth failure occurs. Failed lamps are not replaced. Assume that life x of a lamp has the probability density funciton

$$f(x) = \frac{1}{\theta} e^{-t/\theta}, \; x > 0, \; \theta > 0.$$

Show that the expected waiting time to the rth failure is

$$\theta \sum_{i=1}^{r} \left(\frac{1}{N - i + 1} \right).$$

18. What are three strategies of replacement of items which follow sudden failure mechanisms ? Explain each of them with examples.

19. Explain the O.R. Methodology of solving replacement problems.

EXAMINATIONS REVIEW PROBLEMS

1. A machine shop has a press which is to be replaced as it wears out. A new pess is to be installed now and an optimum replacement plan is to be determined for next 7 years after which the press is no longer required
Following data are availble :

Years	:	1	2	3	4	5	6	7
Cost of New Machine	:	500	525	550	600	650	725	800
Salvage Value	:	250	125	75	50	40	25	0
Operating Cost	:	150	200	250	300	375	450	575

Find an optimum replacement policy.
[**Ans.** At the end of 3rd year.]

2. A company has the option to buy one of the mini computers : MINICOMP and CHIPCOMP. **Minicomp** costs Rs.5 takhs, and running and maintenance costs are Rs. 60,000 for each of the first five years, increasing by Rs. 20,000 per year in the sixth and subsequent years. **Chipcomp** has the same capacity as **Minicomp**, but costs only Rs. 2,50,000. However, its running and maintenace cost are Rs. 1,20,000 per year in the first five years, and increasing by Rs. 20,000 per year thereafter. If the money is worth 10% per year, which computer should be purchased ? What are the optimal replacement periods for each one of the computers ? Assume that there is no salvage value for either computer. Explain your analysis. **[C.A. (May) 90]**
[**Ans.** Rs. 164,186; 9 years and Rs. 173, 195; 7 years.]

3. XYZ organization manufacturing plastic mouldings is considering the replacement of two moulding machines. Capital for machine X initially is Rs.1,80,000 and for machine Y is Rs.2,00,000. For a 6-year period the estimated maintenace costs were established and are presented in the following table :

Moulding X

Year	:	1	2	3	4	5	6
Maintenance Cost (Rs.)	:	12,000	14,000	19,000	22,000	50,000	60,000

Moulding Y

Year	:	1	2	3	4	5	6
Maintenance Cost (Rs.)	:	2,600	4,200	9,000	12,000	25,000	35,000

The department concerned with replacement has stated that it would be advisable, regarding cost, to replace machine X after 4 years and machine Y after 3 years. Assuming interest rate of 10 per cent, use a present worth value as a basis for cost comparison and components on the policy suggested by the department responsible for machine replacement.

4. A company is considering to replace grinder X presently of worth Rs. 10,000 by a new grinder Y of Rs. 20,000 but will be economic in running expenditures. The expected life of grinder X is 5 years with running expenditure of Rs. 4,000 in first year and then additional increase of Rs. 400 per year for next four years. For the new grinder Y, the annual running cost is Rs. 1000 per year and a scrap value of Rs. 2000. As an advisor to the company, find
(*i*) The present value of the cost of old and new grinders considering 12% normal rate of interest.
(*ii*) Suggest whether the old grinder be replaced by the new grinder, assuming the life of new grider to be 5 years.
[Meerut 99]

5. The management of a large hotel is considering the periodic replacement of light bulbs fitted in its rooms. There are 500 rooms in the hotel and each room has 6 bulbs. The management is now following the policy of replacing the bulbs as they fail at a total cost of Rs. 3 per bulb. The management feels that this cost can be reduced to Re. 1 by adopting the periodic replacement method. On the basis of information given below, evaluate the alternative and make a recommendation to the management.

Months of use	:	1	2	3	4	5
Per cent of bulbs failing by that month	:	10	25	50	80	100

6. The cost of a new car is Rs. 10,000. Compare the optimum moment of replacement assuming the following cost informations :

Age of car	Repair cost per year	Salvage value at the end of year
1	5,000	8,000
2	10,000	6,400
3	10,000	5,120

Assume that the repairs are made at the end of each year only if the car is to be retained, and are not necessary if the car is to be sold for its salvage value. Also, assume that the rate of discount is 10%. **[I.A.S. (Main) 90]**

[**Hint.** The discount rate is given $v = \frac{100}{100 + 10} = 0.9091$]

Also assuming the investment of Rs. 10,000 common for all the three alternatives, viz. whether the car is replaced at the end of first year or second year or third year. Present value of repairing cost of the car for the three years is given as under :

Year (n)	Repairing Cost (R_n)	Present Worth Factor (v^n)	Present Value $R_n v^n$	Loss of Capital $C - S(n)$
1	5,000	0.9091	4,545	2000
2	10,000	0.8264	8,264	3600
3	10,000	0.7513	7,513	4880

Now if the car is not replaced, then the present worth of money over three years period are given as below :

Year (n)	Repair Cost $R_n v^n$	Cumulative Replacement Cost $\Sigma R_n v^n$	Loss of Capital $[C - S_{(n)}] v^n$	Total Cost	Total Average Cost
1	4545	4545	1818	6363	6363
2	8264	12809	2974	17601	8800
3	7513	20322	3665	28779	9889

Since average cost is minimum in the first year, the car should be replaced at end of first year.

Note. If the car is replaced, then the present worth of money at the end of respective years will be as follows :

End of year	Repair Cost at 10%	Loss of Capital	Total Cost	Cumulative Total	Average
1	Nil	1,818	1,818	1,818	1,818
2	$5{,}000 \times 0.8264 = 4120$	2,974	7,104	8,922	4,461
3	$10{,}000 \times 0.7513 = 3{,}665$	3,665	11,175	20,097	6,699

Again we notice that the minimum cost occurs at the end of first year.]

7. Give a brief critical account of life testing and estimation technique of Epstein. **[Meerut (Stat.) 95]**

8. What is failure rate ? If the failure distribution R has a density and failure rate $r(t)$, prove that :

$$1 - F(t) = \exp\left[-\int_0^t r(x)\,dx\right].$$

Determine the failure rate for the distribution having the density :

$$f(t) = \frac{1}{\lambda}\exp\left[-\frac{(e^t - 1)}{\lambda} + t\right], \lambda > 0, t \geq 0$$

[Raj. Univ. (M.Phil) 93]

9. Define mean time before failure (MTBF) and show that

$$MTBF = \int_0^\infty R(t)\,dt,$$

where $R(t)$ is the reliabiltty function. **[Raj. Univ. (M. Phil) 92]**

10. Write a short note on reliability of a system. **[Raj. Univ. (M. Phil.) 91]**

11. Discuss inclusion of supplimentary variable technique for reliability models.

12. Show that a series system consisting of n independent components has failure rate equal to the sum of the component failure rates.

14. Determine the failure rate and mean time to system failure for the distribution with density

$$f(t) = \frac{\lambda}{(r-1)!}(\lambda t)^{r-1} e^{-\lambda t}, t \geq 0, \lambda > 0, r > 0.$$

15. If the failure distribution has a density f and failure rate $r(t)$, prove that

$$f(t) = r(t)\exp\left[-\int_0^t r(x)\,dx\right]$$

16. For a machine, the following data are available :

Year	0	1	2	3	4	5	6
Cost of spares (Rs.)	—	200	400	700	1000	1400	1600
Salary of maintenance staff (Rs.)	—	1200	1200	1400	1600	2000	2600
Losses due to breakdown (Rs.)	—	600	800	700	1000	1200	1600
Resale value (Rs.)	12000	6000	3000	1500	800	400	400

Determine the optimum period for replacement of the above machine. **[Delhi (FMCI) 2000]**

17. Find the cost per period of individual replacement policy of an installation of 300 bulbs given in the following :

(i) cost of replacing individual bulb is Rs. 3/-

(ii) conditional probability of failure is given below :

Week No.	:	0	1	2	3	4
Conditional Prob. of Failure	:	0	1/10	1/3	2/3	1

[JNTU (Mech. & Prod.) May 2004]

18. There are two offers of coal handling equipment in a thermal power station.

Offer A : Cost Rs. 20,00,000, Capacity : 20 tons/hr

Block (of five year duration)	I	II	III	IV	V	VI
Operating-cum-maintenance cost per year (in Rs. thousand)	120	130	140	160	190	220
Resale (salvage value in Rs. thousand)	1600	1550	1450	1300	1100	800

Offer B : Cost Rs. 40,00,000, Capacity 300 tons/hr

Block (of five year duration)	I	II	III	IV	V	VI
Operating-cum-maintenance cost per year (in Rs. thousand)	140	145	151	160	172	190
Resale (salvage value in Rs. thousand)	3500	3400	3250	3050	2800	2500

Which offer should be accepted consistent with optimum replacement policy for minimum average annual cost ?

[JNTU (Mech. & Prod.) May 2004]

MODEL OBJECTIVE QUESTION

1. An equipment has been purchased for Rs. 120 and is estimated to have 10 years life and a scrap value of Rs. 20 at the end of life. the book value of the equipment at the end of the sixth year, then the interest rate is 5% (using dedining balance methods) will be

(a) Rs. 40·95. (b) Rs. 51·25. (c) Rs. 55·00. (d) Rs. 59·25.

[IES (Mech.) 2002]

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23

QUEUEING THEORY
(Waiting Line Models)

23.1. INTRODUCTION

In everyday life, it is seen that a number of people arrive at a cinema ticket window. If the people arrive "too frequently" they will have to wait for getting their tickets or sometimes do without it. Under such circumstances, the only alternative is to form a queue, called the ***waiting line,*** in order to maintain a proper discipline. Occasionally, it also happens that the person issuing tickets will have to wait, (*i.e.* remains idle), until additional people arrive. Here the arriving people are called the ***customers*** and the person issuing the tickets is called a ***server.***

Another example is represented by letters arriving at a typist's desk. Again, the letters represent the *customers* and the typist represents the *server.* A third example is illustrated by a machine breakdown situation. A broken machine represents a *customer* calling for the service of a repairman. These examples show that the term *customer* may be interpreted in various number of ways. It is also noticed that a service may be performed either by moving the *server* to the *customer* or the *customer* to the *server.*

Thus, it is concluded that waiting lines are not only the lines of human beings but also the aeroplanes seeking to land at busy airport, ships to be unloaded, machine parts to be assembled, cars waiting for traffic lights to turn green, customers waiting for attention in a shop or supermarket, calls arriving at a telephone switch-board, jobs waiting for processing by a computer, or anything else that require work done on and for it are also the examples of costly and critical delay situations. Further, it is also observed that arriving units may form one line and be serviced through only one station (as in a doctor's clinic), may form one line and be served through several stations (as in a barber shop), may form several lines and be served through as many stations (*e.g.* at check out counters of supermarket).

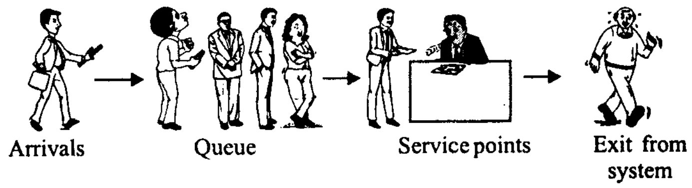

Fig. 23.1 (a). Queueing system with single queue and single service station..

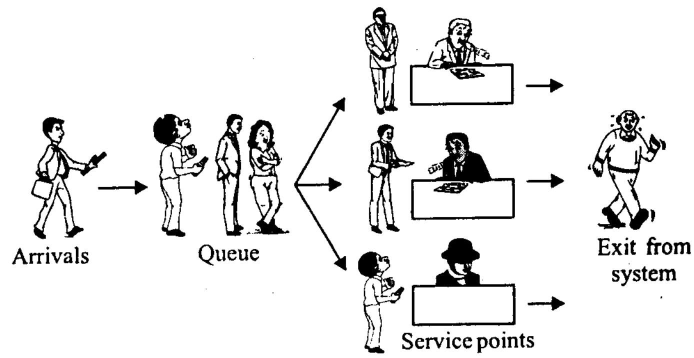

Fig. 23.1 (b). Queueing system with single queue and several service stations.

Servers may be in parallel or in series. When in parallel, the arriving customers may form a single queue as shown in *Fig.* 23.1 or individual queues in front of each server as is common in big post-offices. Service times may be constant or variable and customers may be served singly or in batches (like passengers boarding a bus).

Fig. 23.2 illustrates how a machine shop may be thought of as a system of queues forming in front of a number of service centres, the arrows between the centres indicating possible routes for jobs processed in the shop. Arrivals at a service centre are either new jobs coming into the system or jobs, partially processed, from some other service centre. Departures from a service centre may become the arrivals at another service centre or may leave the system entirely, when processing on these items is complete.

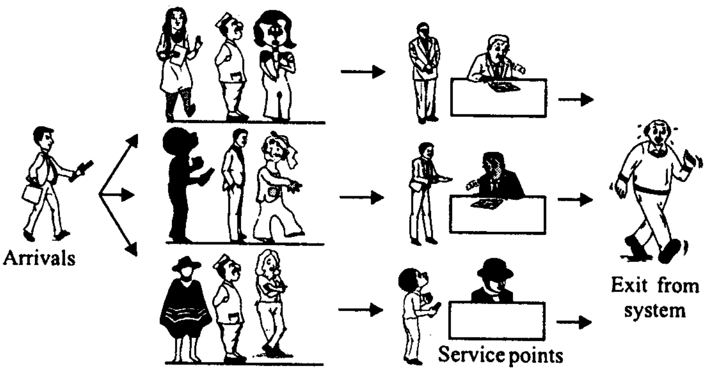

Fig. 23.1 (c). Queueing system with several queues and several service

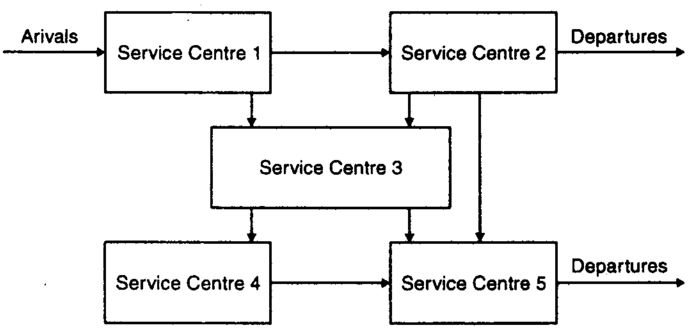

Fig. 23.2. A machine shop as a complex queue.

Queueing theory is concerned with the statistical description of the behaviour of queues with finding, *e.g.*, the probability distribution of the number in the queue from which the mean and variance of queue length and the probability distribution of waiting time for a customer, or the distribution of a server's busy periods can be found. In operational research problems involving queues, investigators must measure the existing system to make an objective assessment of its characteristics and must determine how changes may be made to the system, what effects of various kinds of changes in the system's characteristics would be, and whether, in the light of the costs incurred in the systems, changes should be made to it. A model of the queueing system under study must be constructed in this kind of analysis and the results of queueing theory are required to obtain the characteristics of the model and to assess the effects of changes, such as the addition of an extra server or a reduction in mean service time.

Perhaps the most important general fact emerging from the theory is that the degree of congestion in a queueing system (measured by mean wait in the queue or mean queue length) is very much dependent on the amount of irregularity in the system. Thus congestion depends not just on mean rates at which customers arrive and are served and may be reduced without altering mean rates by regularizing arrivals or service times, or both where this can be achieved.

23.2. QUEUEING SYSTEM

A queueing system can be completely described by

(*a*) the input (or arrival pattern), (*b*) the service mechanism (or service pattern),

(*c*) the 'queue discipline' and (*d*) customer's behaviour.

(a) The input (or arrival pattern). The input describes the way in which the customers arrive and join the system. Generally, the customers arrive in a more or less random fashion which is not worth making the prediction. Thus, the arrival pattern can best be described in terms of probabilities and consequently the probability distribution for inter-arrival times (the time between two successive arrivals) or the distribution of number of customers arriving in unit time must be defined.

The present chapter is only dealt with those queueing systems in which the customers arrive in *'Poisson'* or *'completely random'* fashion (**see** ***sec. 23.7-1***). Other types of arrival pattern may also be observed in practice that have been studied in queueing theory. Two such patterns are observed, where

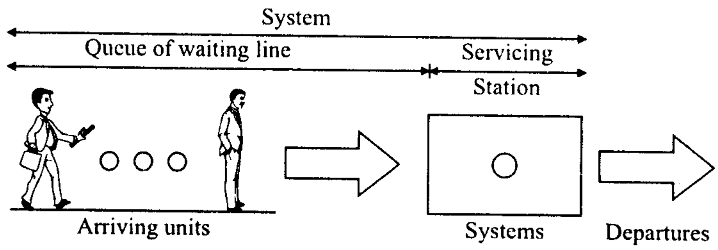

Fig. 23.3. A queueing system with single service station.

(*i*) arrivals are of regular intervals;

(*ii*) there is general distribution (perhaps normal) of time between successive arrivals.

(b) The service mechanism (or service pattern). It is specified when it is known how many customers can be served at a time, what the statistical distribution of service time is, and when service is available. It is true in most situations that service time is a random variable with the same distribution for all arrivals, but cases occur where there are clearly two or more classes of customers (*e.g.* machines waiting repair) each with a different service time distribution. Service time may be constant or a random variable. Distributions of service time which are important in practice are ***'negative exponential distribution'*** and the related ***'Erlang (Gamma) distribution'.*** Queues with the negative exponential service time distribution are studied in the following sections.

In the present chapter, only those queueing systems are discussed in which the service time follows the *'Exponential and Erlang (Gamma)'* probability distributions (see ***sec. 23.7-1*** **to** ***23.7-8***).

(c) **The queue discipline.** The queue discipline is the rule determining the formation of the queue, the manner of the customer's behaviour while waiting, and the manner in which they are chosen for service. The simplest discipline is ***"first come, first served"***, according to which the customers are served in the order of their arrival. For example, such type of queue discipline is observed *at a ration shop, at cinema ticket windows, at railway stations,* etc. If the order is reversed, we have the ***"last come, first served"*** discipline, as in the case of a big godown the items which come last are taken out first. An extremely difficult queue discipline to handle might be ***"service in random order"*** or ***"might is right"***.

Properties of a queueing system which are concerned with waiting times, in general, depend on queue discipline. For example, the variance of waiting time will be much greater with the queue discipline ***'first come, last served'*** than with ***'first come, first served'***, although mean waiting time will remain unaffected.

The following notations are used for describing the nature of service discipline.

FIFO → First In, First Out or ***FCFS*** → First Come, First Served

LIFO → Last In, First Out or ***FILO*** → First In, Last Out.

SIRO → Service in Random Order

This chapter shall be concerned only with the customers which are served in the order in which they arrive at the service facility, that is, ***'first come, first served'*** discipline.

(d) Customer's behaviour. The customers generally behave in four ways :

(*i*) **Balking.** A customer may leave the queue because the queue is too long and he has no time to wait, or there is not sufficient waiting space.

(*ii*) **Reneging.** This occurs when a waiting customer leaves the queue due to impatience.

(*iii*) **Priorities.** In certain applications some customers are served before others regardless of their order of arrival. These customers have *priority* over others.

(*iv*) **Jockeying.** Customers may *jockey* from one waiting line to another. It may be seen that this occurs in the supermarket.

(e) Size of a Population : The collection of potential customers may be very large or of a moderate size. In a railway booking counter the total number of potential passengers is so large that although theoretically finite it can be regarded as infinity for all practical purposes. The assumption of infinite population is very convenient for analysing a queuing model. Howerver, this assumption is not valid where the customer group is represented by few machines in workshop that require operator facility from time to time. If the population size is finite then the analysis of queueing model becomes more involved.

(f) Maximum Length of a Queue : Sometimes only a finite number of customers are allowed to stay in the system although the total number of customers in the population may or may not be finite. For example, a doctor may have appointments with k patients in a day. If the number of patients asking for appointment exceeds k, they are not allowed to join the queue. Thus, although the size of the population is infinite, the maximum number permissible in the system is k.

Q. 1. Explain briefly the main characteristics of queueing system.
2. Describe the fundamental components of a queueing process and give suitable examples. **[C.A. (Nov) 92]**
3. List the factors that constitute the basic elements of a queueing model. For each of these enumerate the alternatives posible. Represent this diagramatically to cover all possible implimentations of a queueing model. **[IGNOU 99 (Dec.)]**
4. What is queueing theory ? **[JNTU (MCA III) 2004]**

23.3. QUEUEING PROBLEM

In a specified queueing system, the problem is to determine the following :

(a) Probability distribution of queue length. When the nature of probability distributions of the arrival and service patterns is given, the probability distribution of queue length can be obtained. Further, we can also estimate the probability that there is no queue.

(b) Probability distribution of waiting time of customers. We can find the time spent by a customer in the queue before the commencement of his service which is called his ***waiting time.*** The total time spent by him in the system is the waiting time plus service time.

(c) The busy period distribution. We can estimate the probability distribution of busy periods. If we suppose that the server is free initially and customer arrives, he will be served immediately. During his service time, some more customers will arrive and will be served in their turn. This process will continue in this way until no customer is left unserved and the server becomes free again. Whenever this happens, we say that a **busy period** has just ended. On the other hand, during **idle periods** no customer is present in the system. A busy period and the idle period following it together constitute a *busy cycle*. The study of the busy period is of great interest in cases where technical features of the server and his capacity for continuous operations must be taken into account.

23.4. TRANSIENT AND STEADY STATES

Queueing theory analysis involves the study of a system's behaviour over time. ***A system is said to be in "transient state" when its operating characteristics (behaviour) are dependent on time.*** This usually occurs at the early stages of the operation of the system where its behaviour is still dependent on the initial conditions. However, since we are mostly interested in the *"long run"* behaviour of the system, mainly the attention has been paid toward *"steady state"* results.

A steady state condition is said to prevail when the behaviour of the system becomes independent of time. Let $P_n(t)$ denote the probability that there are n units in the system at time t. In fact, the change of $P_n(t)$ with respect to t is described by the derivative $[dP_n(t)/dt]$ or $P_n'(t)$. Then the queueing system is said to become *'stable'* eventually, in the sense that the probability $P_n(t)$ is independent of time, that is, remains the same as time passes ($t \to \infty$). Mathematically, in steady state

$$\lim_{t\to\infty} P_n(t) = P_n \text{ (independent of } t) \Rightarrow \lim_{t\to\infty} \frac{dP_n(t)}{dt} = \frac{dP_n}{dt} \Rightarrow \lim_{t\to\infty} P_n'(t) = 0.$$

In some situations, if the arrival rate of the system is larger than its service rate, a steady state cannot be reached regardless of the length of the elapsed time. In fact, in this case the queue length will increase with time and theoretically it could build up to infinity. Such case is called the ***"explosive state"***.

In this chapter, only the steady state analysis will be considered. We shall not treat the *'transient'* and *'explosive'* states.

Q. 1. What is queueing problem ? Explain queueing system, transient and steady state. **[Garhwal M.Sc. (Stat.) 96]**

2. What is a queueing theory problem ? Describe the advantages of queueing theory to a business executive with a view to persuading him to make use of the same in management. **[Garhwal M.Sc. (Stat.) 95]**

3. What do you understand by a queue ? Give some important applications of queueing theory. **[Garhwal M.Sc. (Stat.) 92, 91]**

4. Write an essay on various characteristics of a queueing system. **[Virbhadra 2000]**

5. Discuss the stationery state of the queue system. **[JNTU (Mech. & Prod.) 2004]**

23.5. A LIST OF SYMBOLS

Unless otherwise stated, the following symbols and terminology will be used henceforth in connection with the queueing models. The reader is reminded that a queueing system is defined to include the *queue* and the *service stations* both. (see *Fig.* 23.3).

n = number of units in the system

$P_n(t)$ = transient state probability that exactly n calling units are in the queueing system at time t

E_n = the state in which there are n calling units in the system

P_n = steady state probability of having n units in the system

λ_n = mean arrival rate (expected number of arrivals per unit time) of customers (when n units are present in the system)

μ_n = mean service rate (expected number of customers served per unit time when there are n units in the system)

λ = mean arrival rate when λ_n is constant for all n

μ = mean service rate when μ_n is constant for all $n \geq 1$

s = number of parallel service stations

$\rho = \lambda/\mu s$ = traffic intensity (or utilization factor) for servers facility, that is, the expected fraction of time the servers are busy

$\phi_T(n)$ = probability of n services in time T, given that servicing is going on throughout T

Line length (or queue size)
= number of customers in the queueing system

Queue length
= line length (or queue size)–(number of units being served)

$\Psi(w)$ = probability density function (p.d.f.) of waiting time in the system

L_s = expected line length, *i.e.*, expected number of customers in the *system*

L_q = expected queue length, *i.e.*, expected number of customers in the *queue*

W_s = expected waiting time per customer in the system

W_q = expected waiting time per customer in the queue

$(W \mid W > 0)$ = expected waiting time of a customer who has to wait

$(L \mid L > 0)$ = expected length of non-empty queues, *i.e.*, expected number of customers in the queue when there is a queue

$P(W > 0)$ = probability of a customer having to wait for service

$\binom{n}{r}$ = the binomial coefficient ${}^nC_r$.

$$= \frac{n\,!}{r\,!\,(n-r)\,!} = \frac{n\,(n-1)\,\ldots\,(n-r+1)}{r\,!} \quad \text{for } r \text{ and } n \text{ non-negative integers } (r \leq n).$$

23.6. TRAFFIC INTENSITY (OR UTILIZATION FACTOR)

An important measure of a simple queue ($M \mid M \mid 1$) is its *traffic intensity,* where

$$\text{Traffic intensity } (\rho) = \frac{\text{mean arrival rate}}{\text{mean service rate}} = \frac{\lambda}{\mu}$$

i.e.,
$$\rho = \frac{1/\mu}{1/\lambda} = \frac{\text{Mean service time}}{\text{Mean inter–arrival time}}$$

The unit of traffic intensity is *Erlang*.

Here it should be noted carefully that a necessary condition for a system to have settled down to steady state is that $\rho < 1$ or $\lambda/\mu < 1$ or $\lambda < \mu$, *i.e., arrival rate < service rate.*

If this is not so, *i.e.*, $\rho > 1$, the arrival rate will be greater than the service rate and consequently, the number of units in the queue tends to increase indefinitely as the time passes on, provided the rate of service is not affected by the length of queue.

23.7. PROBABILITY DISTRIBUTIONS IN QUEUEING SYSTEMS

The arrival pattern of customers at a queueing system varies between one system and another, but one pattern of common occurrence in practice, which turns out to be relatively easy to deal with mathematically, is that of ***'completely random arrivals'***. This phrase means something quite specific, and we discuss what does it mean before dealing in the subsequent sections with a variety of queueing systems. In particular, we show that, if arrivals are *'completely random'*, the number of arrivals in unit time has a *Poisson distribution*, and the intervals between successive arrivals are distributed *negative exponentially*.

23.7-1. Distribution of Arrivals 'The Poisson Process' (Pure Birth Process)

In many situations the objective of an analysis consists of merely observing the number of customers that enter the system. The model in which only arrivals are counted and no departures take place are called *pure birth models*. The term *'birth'* refers to the arrival of a new calling unit in the system, and the *'death'* refers to the departure of a served unit. As such *pure birth* models are not of much importance so far as their applicability to real life situation is concerned, but these are very important in the understanding of completely random arrival problems. **[Bhubneshwar (IT) 2004]**

Theorem 23.1. (***Arrival Distribution Theorem***). *If the arrivals are completely random, then the probability distribution of number of arrivals in a fixed time-interval follows a Poisson distribution.*

[Agra 99; Meerut (Stat.) 98; Garhwal M.Sc. (Math.) 94; Raj. Univ. (Math) 93]

Proof. In order to derive the arrival distribution in queues, we make the following three assumptions (sometimes called the ***axioms***).

1. Assume that there are n units in the system at time t, and the probability that exactly one arrival (birth) will occur during small time interval Δt be given by $\lambda \Delta t + O(\Delta t)$, where λ is the arrival rate independent of t and $O(\Delta t)$ includes the terms of higher order of Δt.
2. Further assume that the time Δt is so small that the probability of more than one arrival in time Δt is $O(\Delta t)^2$, *i.e.*, almost zero.
3. The number of arrivals in non-overlapping intervals are statistically independent, *i.e.*, the process has independent increments.

We now wish to determine the probability of n arrivals in a time interval of length t, denoted by $P_n(t)$. Clearly, n will be an integer greater than or equal to zero. To do so, we shall first develop the differential-difference equations governing the process in two different situations.

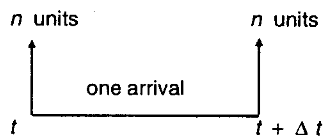

Fig. 23.4.

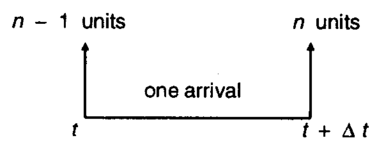

Fig. 23.5.

Case I. ***When*** $n > 0$**.** For $n > 0$, there may be two mutually exclusive ways of having n units at time $t + \Delta t$.

(*i*) There are n units in the system at time t and no arrival takes place during time interval Δt. Hence, there will be n units at time $t + \Delta t$ also. This situation is better explained in *Fig. 23.4.*

Therefore, the probability of these two combined events will be

$$= \text{Prob. of } n \text{ units at time } t \times \text{Prob. of no arrival during } \Delta t = P_n(t) \,.\, (1 - \lambda\Delta t) \qquad ...(23.1)$$

[since prob. of exactly one arrival in $\Delta t = \lambda\Delta t$, prob. of no arrival becomes $= 1 - \lambda\Delta t$.]

(*ii*) *Alternately*, there are $(n - 1)$ units in the system at time t, and one arrival takes place during Δt. Hence there will remain n units in the system at time $t + \Delta t$. This situation is better explained in *Fig. 23.5*.
Therefore, the probability of these two combined events will be

$$= \text{Prob. of } (n - 1) \text{ units at time } t \times \text{Prob. of one arrival in time } \Delta t = P_{n-1}(t) . \lambda\Delta t \qquad ...(23.2)$$

Note. Since the probability of more than one arrival in Δt is assumed to be negligible, other alternatives do not exist.

Now, adding above two probabilities [given by (23.1) and (23.2)], we get the probability of n arrivals at time $t + \Delta t$, *i.e.*

$$P_n(t + \Delta t) = P_n(t)\,(1 - \lambda\Delta t) + P_{n-1}(t)\,\lambda\Delta t \qquad ...(23.3)$$

Case 2. *When n* = 0.

$$P_0(t + \Delta t) = \text{Prob. [}\textit{no unit at time t}\text{]} \times \text{Prob. [}\textit{no arrival in time } \Delta t\text{]}$$

$$\therefore \qquad P_0(t + \Delta t) = P_0(t)\,(1 - \lambda\Delta t). \qquad ...(23.4)$$

Rewriting the equations (23.3) and (23.4) after transposing the terms $P_n(t)$ and $P_0(t)$ to left hand sides, respectively, we get

$$P_n(t + \Delta t) - P_n(t) = P_n(t)\,(-\lambda\Delta t) + P_{n-1}(t)\,\lambda\Delta t, \qquad n > 0 \qquad ...(23.3)'$$

$$P_0(t + \Delta t) - P_0(t) = P_0(t)\,(-\lambda\Delta t) \qquad n = 0 \qquad ...(23.4)'$$

Dividing both sides by Δt and then taking limit as $\Delta t \to 0$,

$$\lim_{\Delta t \to 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda\, P_{n-1}(t) \qquad ...(23.5)$$

$$\lim_{\Delta t \to 0} \frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda\, P_0(t) \qquad ...(23.6)$$

Since by definition of first derivative, $\lim\limits_{\Delta t \to 0} \dfrac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \dfrac{d\,P_n(t)}{dt} = P_n'(t)$,

the equations (23.6) and (23.5) respectively can be written as

$$P_0'(t) = -\lambda P_0(t), \qquad n = 0 \qquad ...(23.7)$$

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \qquad n > 0 \qquad ...(23.8)$$

This is known as the *system of differential-difference equations.*

To solve the equations (23.7) and (23.8) by iterative method :

Equation (23.7) can be written as

$$\frac{P_0'(t)}{P_0(t)} = -\lambda \;\text{ or }\; \frac{d}{dt}[\log P_0(t)] = -\lambda \qquad ...(23.9)$$

Integrating both sides w.r.t. 't',

$$\log P_0(t) = -\lambda t + A \qquad ...(23.10)$$

The constant of integration can be determined by using the boundary conditions :

$$P_n(0) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n > 0 . \end{cases}$$

Substituting $t = 0$, $P_0(0) = 1$ in (23.10), find $A = 0$. Thus, (23.10) gives

$$\log P_0(t) = -\lambda t \;\text{ or }\; P_0(t) = e^{-\lambda t} \qquad ...(23.11)$$

Putting $n = 1$ in (23.8), $\qquad P_1'(t) = -\lambda P_1(t) + \lambda P_0(t)$

or $\qquad P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}. \qquad ...(23.12)$

Since this is the linear differential equation of first order, it can be easily solved by multiplying both sides of this equation by the integrating factor, I.F. $= e^{\int \lambda dt} = e^{\lambda t}$.

Thus, eqn. (23.12) becomes

$$e^{\lambda t}\,[P_1'(t) + \lambda P_1(t)] = \lambda \;\text{ or }\; \frac{d}{dt}[e^{\lambda t}\,P_1(t)] = \lambda$$

Now integrating both sides w.r.t. 't'

$$e^{\lambda t} P_1(t) = \lambda t + B, \qquad ...(23.13)$$

where B is the constant of integration.

In order to determine the constant B, put $t = 0$ in (23.13), and get

$$P_1(0) = 0 + B \text{ or } B = 0 \qquad [\because P_1(0) = 0]$$

Substituting $B = 0$ in (23.13), $\quad P_1(t) = \dfrac{\lambda t\, e^{-\lambda t}}{1\,!} \qquad ...(23.14)$

Similarly, putting $n = 2$ in (23.8) and using the result (23.14), we get the equation

$$P_2'(t) + \lambda P_2(t) = \lambda \frac{(\lambda t)\, e^{-\lambda t}}{1\,!} \quad \text{or} \quad \frac{d}{dt}[e^{\lambda t} P_2(t)] = \frac{\lambda\,(\lambda t)}{1\,!}.$$

Integrating w.r.t. 't' $\quad e^{\lambda t} P_2(t) = \dfrac{(\lambda t)^2}{2\,!} + C,$

Put $t = 0$, $P_2(0) = 0$ to obtain $C = 0$. Hence

$$P_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2\,!}, \text{ for } n = 2 \qquad ...(23.15)$$

Similarly, obtain $\quad P_3(t) = \dfrac{(\lambda t)^3 e^{-\lambda t}}{3\,!}$, for $n = 3$

Likewise, in general, $\quad P_m(t) = \dfrac{(\lambda t)^m e^{-\lambda t}}{m\,!}$ for $n = m$. $\qquad ...(23.16)$

If, anyhow, it can be proved that the result (23.16) is also true for $n = m + 1$, then by induction hypothesis result (23.16) will be true for general value of n.

To do so, put $n = m + 1$ in (23.8) and get

$$P'_{m+1}(t) + \lambda P_{m+1}(t) = \lambda \frac{(\lambda t)^m e^{-\lambda t}}{m\,!} \qquad \text{[using the results (23.16)]}$$

or

$$\frac{d}{dt}[e^{\lambda t} P_{m+1}(t)] = \frac{(\lambda t)^m (\lambda)}{m\,!}.$$

Integrating both sides, $\quad e^{\lambda t} P_{m+1}(t) = \dfrac{(\lambda t)^{m+1}}{(m+1)\, m\,!} + D,$

Again, putting $t = 0$, $P_{m+1}(0) = 0$, we get $D = 0$. Therefore,

$$\therefore \quad P_{m+1}(t) = \frac{(\lambda t)^{m+1} e^{-\lambda t}}{(m+1)\,!}.$$

Hence, in general, $\quad \boldsymbol{P_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n\,!}}, \qquad ...(23.17)$

which is a ***Poisson distribution formula.*** This completes the proof of the theorem.

Note. After carefully understanding the above procedure, the students can much reduce the number of steps by solving the differential equation of the standard from : $y' + P(x)\, y = Q(x)$, using the formula

$$y \,.\, e^{\int P dx} = \int Q(x)\, (e^{\int P dx})\, dx + C,$$

where $e^{\int P dx}$ is the integrating factor (I.F.)

Alternative Method : *Generating Function Technique.*

The system of equations (23.7) and (23.8) is

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \qquad n > 0 \qquad ...(i)$$

$$P_0'(t) = -\lambda P_0(t), \qquad n = 0 \qquad ...(ii)$$

We define the generating function of $P_n(t)$ as, $P(z, t) = \sum_{n=0}^{\infty} P_n(t)\, z^n$. Also, $P'(z, t) = \sum_{n=0}^{\infty} P_n'(t)\, z^n$.

Multiplying both sides of (i) by z^n and taking summation for $n = 1, 2, ..., \infty$, we get

$$\sum_{n=1}^{\infty} z^n P_n'(t) = -\lambda \sum_{n=1}^{\infty} z^n P_n(t) + \lambda \sum_{n=1}^{\infty} P_{n-1}(t) z^n \qquad \text{...(iii)}$$

Now adding (ii) and (iii), we get

$$\sum_{n=0}^{\infty} z^n P_n'(t) = -\lambda \sum_{n=0}^{\infty} z^n P_n(t) + \lambda \sum_{n=0}^{\infty} z^{n+1} P_n(t)$$

or
$$P'(z, t) = -\lambda P(z, t) + \lambda z P(z, t) \text{ or } \frac{P'(z, t)}{P(z, t)} = \lambda (z - 1)$$

or
$$\frac{d}{dt}[\log P(z, t)] = \lambda (z - 1)$$

Integrating both sides, $\log P(z, t) = \lambda (z - 1) t + E.$ ...(iv)

To determine E, we put $t = 0$ to get $\log P(z, 0) = E$

But, $P(z, 0) = \sum_{n=0}^{\infty} z^n P_n(0) = P_0(0) + \sum_{n=1}^{\infty} z^n P_n(0)$

$= 1 + 0 = 1$ ($\because P_0(0) = 1$, and $P_n(0) = 0$ for $n > 0$)

Therefore, $E = \log P(z, 0) = \log 1 = 0.$

$\therefore$ eqn. (iv) becomes, $\log P(z, t) = \lambda(z - 1) t$ or $P(z, t) = e^{\lambda (z-1) t}$.

Now, $P_n(t)$ can be defined as $P_n(t) = \frac{1}{n!}\left[\frac{d^n P(z, t)}{dz^n}\right]_{t=0}$

Using this formula, $P_0(t) = [P(z, t)]_{z=0} = e^{-\lambda t}$

$$P_1(t) = \left[\frac{d P(z, t)}{dz}\right]_{t=0} = [e^{\lambda (z-1) t} \lambda t]_{z=0} = \frac{e^{-\lambda t} \lambda t}{1!}$$

$$P_2(t) = \frac{1}{2!}\left[\frac{d^2 P(z, t)}{dz^2}\right]_{t=0} = \frac{(\lambda t)^2 e^{-\lambda t}}{2!}$$

... ...

In general, $$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}. \qquad \text{...(v)}$$

Thus the probability of n arrivals in time 't' follows the ***Poisson law*** given by eqn. (v).

Q. 1. Show that 'n' the number of arrivals in a queue in time t follows the Poisson distribution, stating the assumptions clearly.
2. Show that the distribution of the number of births up to time 'T' in a simple birth process follows the Poisson law.
3. What do you understand by a queue ? Give some applications of queueing theory.
4. Explain what do you mean by Poisson process. Derive the Poisson distribution, given that the probability of single arrival during a small time interval Δt is $\lambda \Delta t$ and that of more than one arrival is negligible. **[JNTU (B. Tech.) 2002; Meerut (Maths.) 96]**
5. State when a model is called Pure Birth Process in Queueing Theory. **[Bhubneshwar (IT) 2004]**

23.7-2. Properties of Poisson Process of Arrivals

It has already been derived that–*if n be the number of arrivals during time interval t, then the law of probability in Poisson process is given by*

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, ..., \infty \qquad \text{...(23.18)}$$

where λt is the parameter.

(1) Since mean $E(n) = \lambda t$, and var. $(n) = \lambda t$, ...(23.19)

the average (expected) number of arrivals in unit time will be

$$E(n)/t = \lambda = \text{ mean arrival rate (or input rate).}$$

(2) If we consider the time interval $(t, t + \Delta t)$, where Δt is sufficiently small, then

$$P_0(\Delta t) = \text{Prob } [\textit{no arrival in time } \Delta t]$$

Putting $n = 0$ and $t = \Delta t$ in (23.18)

$$P_0(\Delta t) = \frac{e^{-\lambda\Delta t}}{0\,!} = e^{-\lambda\Delta t} = 1 - \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2\,!} - \ldots = 1 - \lambda\Delta t + O(\Delta t)$$

where the term $O(\Delta t)$ indicates a quantity that is negligible compared to Δt. More precisely, $O(\Delta t)$ represents any function of Δt such that

$$\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0.$$

[For example, $(\Delta t)^2$ can be replaced by $O(\Delta t)$ because $\lim_{\Delta t \to 0} \frac{(\Delta t)^2}{\Delta t} = 0$. This notation will be very useful for summarizing the negligible terms which do not enter in the final result]

$$\therefore \qquad P_0(\Delta t) = 1 - \lambda\Delta t \qquad ...(23.20)$$

which means that the probability of no arrival in Δt is $1 - \lambda\Delta t$. In the similar fashion, $P_1(\Delta t)$ can be written as

$$P_1(\Delta t) = \frac{(\lambda\Delta t)\, e^{-\lambda\Delta t}}{1\,!} \qquad \text{[putting } n = 1, t = \Delta t \text{ in (23.18)]}$$

$$= \lambda\Delta t \left[1 - \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2\,!} - \ldots\right] \qquad = \lambda\Delta t + O(\Delta t).$$

Neglecting the term $O(\Delta t)$, $\qquad P_1(\Delta t) = \lambda\Delta t$, $\qquad$ ...(23.21)

which means that the probability of one arrival in time Δt is $\lambda\Delta t$.

Similarly, $\qquad P_2(\Delta t) = \dfrac{(\lambda\Delta t)^2\, e^{-\lambda\Delta t}}{2\,!} = (\lambda\Delta t)^2 \left[1 - \lambda\Delta t + \dfrac{(\lambda\Delta t)^2}{2\,!} - \ldots\right] = O(\Delta t).$

Again neglecting the term $O(\Delta t)$, we have $\qquad P_2(\Delta t) = 0$, $\qquad$ ...(23.22)

ans so on. Thus, it is concluded from the property of Poisson process that the probability of more than one arrival in time Δt is negligibly small, provided the terms of second and higher order of Δt are considered to be negligibly small. Symbolically,

$$P_n(\Delta t) = \text{ negligibly small for all } n > 1. \qquad ...(23.23)$$

Q. State some important properties of Poission's process. **[JNTU (B. Tech.) 2003]**

23.7-3. Distribution of Inter-Arrival Times (Exponential Process)

Let T be the time between two consecutive arrivals (called the inter-arrival time), and $a(T)$ denotes the probability density function of T. Then the following important theorem can be proved.

Theorem 23.2. *If n, the number of arrivals in time t, follows the Poisson distribution,*

$$P_n(t) = (\lambda t)^n\, e^{-\lambda t}/n\,!\,, \qquad ...(23.24)$$

then T (the inter-arrival time) obeys the negative exponential law

$$a(T) = \lambda e^{-\lambda T} \qquad ...(23.25)$$

and vice-versa. **[Kanpur 2000; Garhwal M.Sc. (Stat.) 95; Raj .Univ. (M.Phil) 91]**

Proof. Suppose that $t_0 =$ instant of an arrival initially.

Since there is no arrival in the intervals $(t_0, t_0 + T)$ and $(t_0 + T, t_0 + T + \Delta T)$, therefore $(t_0 + T + \Delta T)$ will be the instant of subsequent arrival.

Therefore, putting $t = T + \Delta T$ and $n = 0$ in (5.24),

$$P_0(T + \Delta T) = \frac{[\lambda(T + \Delta T)]^0 \,.\, e^{-\lambda(T+\Delta T)}}{0\,!} = e^{-\lambda(T+\Delta T)}$$

$$= e^{-\lambda T} \,.\, e^{-\lambda\Delta T} = e^{-\lambda T}\ [1 - \lambda\Delta T + O(\Delta T)]$$

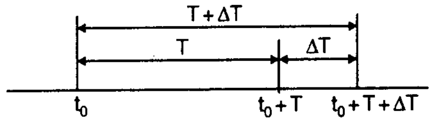

Fig. 23.6

Since $P_0(T) = e^{-\lambda T}$ from (23.24),

$$P_0(T + \Delta T) = P_0(T)\ [1 - \lambda\Delta T + O(\Delta T)]$$

or $\qquad P_0(T + \Delta T) - P_0(T) = P_0(T)\ [-\lambda\Delta T + O(\Delta T)].$

Dividing both sides by ΔT,

$$\frac{P_0(T+\Delta T)-P_0(T)}{\Delta T}=-\lambda P_0(T)+\frac{O(\Delta T)}{\Delta T}P_0(T)$$

Now taking limit on both sides as $\Delta T \to 0$,

$$\lim_{\Delta T\to 0}\frac{P_0(T+\Delta T)-P_0(T)}{\Delta T}=\lim_{\Delta T\to 0}\left[-\lambda P_0(T)+\frac{O(\Delta T)}{\Delta T}P_0(T)\right]$$

or
$$\frac{dP_0(T)}{dT}=-\lambda P_0(T).\left[\text{since }\lim_{\Delta T\to 0}\frac{O(\Delta T)}{\Delta T}=0\right] \qquad ...(23.26)$$

But, L.H.S. of (23.26) is denoting the *probability density function* of T, say $a(T)$. Therefore,

$$*a(T)=**\lambda P_0(T). \quad \textbf{(see footnote)} \qquad ...(23.27)$$

But, from equation (23.24), $P_0(T)=e^{-\lambda T}$. Putting this value of $P_0(T)$ in (23.27),

$$a(T)=\lambda e^{-\lambda T} \qquad ...(23.28)$$

which is the *exponential law of probability* for T with mean $1/\lambda$ and variance $1/\lambda^2$, *i.e.*,

$$E(T)=1/\lambda,\ \text{Var.}\ (T)=1/\lambda^2.$$

In a similar fashion, the converse of this theorem can be proved.

Q. 1. Give the axioms characterizing a Poisson process. If the number of arrivals in some time interval follows a Poisson distribution, show that the distribution of the time interval between two consecutive arrivals is exponential.
[Delhi M.A/M.Sc. (Stat.) 95; Raj. Univ. (M. Phil) 91]

2. Show that if the inter-arrival times are negative exponentially distributed, the number of arrivals in a time period is a Poisson process and conversely.

3. If the intervals between successive arrivals are i.i.d. random variables which follow the negative exponential distribution with mean $1/\lambda$, then show that the arrivals form a Poisson Process with mean λt. **[Garhwal M.Sc. (Stat.) 91]**

4. Show that inter-arrival times are distributed exponentially, if arrival is a Poisson process. Prove the converse also.
[Delhi M.A/M.Sc (OR) 92.]

5. State the three axioms underlying the exponential process. Under exponential assumptions can two events occur during a very small interval. **[Meerut 2002]**

23.7-4. Markovian Property of Inter-arrival Times

Statement. *The Markovian property of inter-arrival times states that at any instant the time until the next arrival occurs is independent of the time that has elapsed since the occurrence of the last arrival. That is to say,*

Prob. $[T \ge t_1 \mid T \ge t_0] =$ *Prob.* $[0 \le T \le t_1 - t_0]$

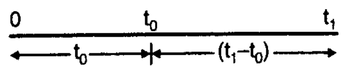

Fig. 23.7

Proof. Consider

$$\text{Prob. }[T\ge t_1 \mid T\ge t_0]=\frac{\text{Prob. }[(T\ge t_1)\text{ and }(T\ge t_0)]}{\text{Prob. }[T\ge t_0]}\ \text{(formula of conditional probability)} \qquad ...(23.29)$$

Since the inter-arrival times are exponentially distributed, the right hand side of equation (23.29) can be written as

$$\frac{\int_{t_0}^{t_1}\lambda e^{-\lambda t}\,dt}{\int_{t_0}^{\infty}\lambda e^{-\lambda t}\,dt}=\frac{e^{-\lambda t_1}-e^{-\lambda t_0}}{-e^{-\lambda t_0}}$$

$$\therefore \qquad \text{Prob. }[T\ge t_1 \mid T\ge t_0]=1-e^{-\lambda(t_1-t_0)} \qquad ...(23.30)$$

---

* According to probability distributions $d/dx\,[F(x)]=f(x)$, where $F(x)$ is the *'distribution function'* and $f(x)$ is the *'probability density function'*. Hence by the similar argument, we may write $d/dT\,[P_0(T)]=a(T)$, where $P_0(T)$ is the probability distribution function for no arrival in time T, and $a(T)$ is denoting the corresponding probability density function of T.

** Since *'probability density function'* is always non-negative, so neglect the negative sign from right side of equation (23.26).

But, $\qquad \text{Prob. } [0 \le T \le t_1 - t_0] = \int_0^{t_1 - t_0} \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda(t_1 - t_0)}.$ ...(23.31)

Thus, by virtue of equations (23.30) and (23.31), it can be concluded that

$$\text{Prob. } [T \ge t_1 \mid T \ge t_0] = \text{Prob. } [0 \le T \le t_1 - t_0].$$

This proves the *Markovian property* of inter-arrival times.

Q. State and prove the Markovian property of inter-arrival times (*i.e.* of exponential distribution).

23.7-5. Distribution of Departures (or Pure Death Process)

In this process assume that there are N customers in the system at time $t = 0$. Also, assume that no arrivals (births) can occur in the system. Departures occur at a rate μ per unit time, *i.e.*, output rate is μ. We wish to derive the distribution of departures from the system on the basis of the following three *axioms* :

(1) Prob. [*one departure during* Δt] $= \mu\Delta t + O(\Delta t)^2 = \mu\Delta t$ [$\because$ $O(\Delta t)^2$ is negligible]

(2) Prob. [*more than one departure during* Δt] $= O(\Delta t)^2 \doteq 0$.

(3) The number of departures in non-overlapping intervals are statistically independent and identically distributed random variable, *i.e.*, the process $N(t)$ has independent increments.

First obtain the differential difference equation in three mutually exclusive ways :

Case I. ***When*** $0 < n < N$. Proceeding exactly as in the *Pure Birth Process,*

$$P_n(t + \Delta t) = P_n(t)\,[1 - \mu\Delta t] + P_{n+1}(t)\,\mu\Delta t \qquad ...(23.32)$$

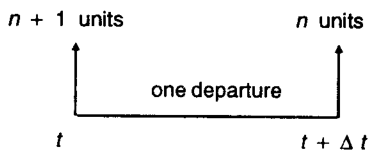

Fig. 23.8

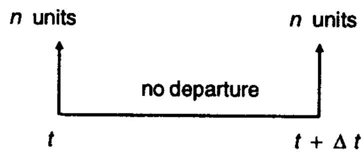

Fig. 23.9

Case II. ***When*** $n = N$. Since there are exactly N units in the system, $P_{n+1}(t) = 0$,

$$\therefore \qquad P_N(t + \Delta t) = P_N(t)\,[1 - \mu\Delta t] \qquad ...(23.33)$$

Case III. ***When*** $n = 0$.

$$P_0(t + \Delta t) = P_0(t) + P_1(t)\,\mu\Delta t \qquad ...(23.34)$$

Since there is no unit in the system at time t, the question of any departure during Δt does not arise. Therefore, probability of no departure is unity in this case.

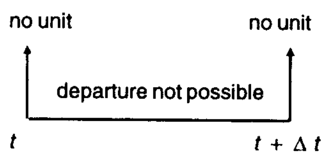

Fig. 23.10.

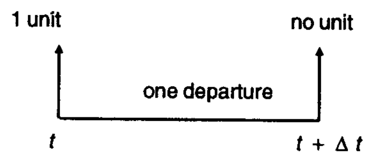

Fig. 23.11.

Now, re-arranging the terms and dividing by Δt, and also taking the limit $\Delta t \to 0$ the equations (23.33), (23.32) and (23.34), respectively, become

$$P_N'(t) = -\mu P_N(t), \qquad n = N \qquad ...(23.35)$$

$$P_n'(t) = -\mu P_n(t) + \mu P_{n+1}(t), \qquad 0 < n < N \qquad ...(23.36)$$

$$P_0'(t) = \mu P_1(t), \qquad n = 0 \qquad ...(23.37)$$

To solve the system of equations (23.35), (23.36) and (23.37) :

Iterative method can be used to solve the system of three equations.

Step 1. From equation (23.35) obtain

$$\frac{P_N'(t)}{P_N(t)} = -\mu \text{ or } \frac{d}{dt} \log P_N(t) = -\mu$$